6 Sturm Liouville Problems

6.1 General comments

In this section, we consider homogeneous linear boundary value problems of the type

$$-(p(x)y')' + q(x)y = \lambda y, \quad B_a[y] = 0, \quad B_b[y] = 0,$$

where $p(x), q(x) : [a, b] \to [0, \infty)$ are continuous functions, $p'(x)$ is continuous, and $\lambda \in \mathbb{R}$. As in the previous section we have boundary conditions of the form

$$B_a[y] = \alpha_1 y(a) + \beta_1 y'(a) \quad \text{and} \quad B_b[y] = \alpha_2 y(b) + \beta_2 y'(b). \tag{6.47}$$

Such problems are called **Sturm-Liouville** problems and their solutions have a rich structure as we shall see.

Example 6.1. *The problem*

$$-y''(x) = \lambda y(x), \quad y(0) = y(L) = 0$$

is a simple example of a S-L problem corresponding to the choice $p(x) = 1$, $q(x) = 0$.

Before studying such equations in detail, we must first recall some details about inner product spaces.

6.2 Inner product spaces

Let V be a vector space over the field $\mathbb{R}$.

Definition 6.2. *A map* $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ *is called an* **inner product** *on* V *if for all* $u, v, w \in V$ *and* $\alpha \in \mathbb{R}$
(i) $\langle v, v\rangle \geq 0$ *and* $\langle v, v\rangle = 0$ *if and only if* $v = 0$;
(ii) $\langle u, v\rangle = \langle v, u\rangle$;
(iii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$;
(iv) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$.

Example 6.3. *If* $V = \mathbb{R}^{\mathbf{N}}$, *we have an inner product*

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \right\rangle = x_1y_1 + x_2y_2 + \cdots + x_Ny_N.$$

Definition 6.4. *An* **inner product space** *is a vector space equipped with an inner product.*

Example 6.5. *Let* V *be the space of piecewise continuous functions* $[a, b] \to \mathbb{R}$. *Then we can equip* V *with an inner product*

$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)\,dx.$$

Definition 6.6. *Let V be an inner product space. We say that*
(i) u and v are **orthogonal** *if $\langle u, v\rangle = 0$;*
(ii) S is an **orthogonal set** *of vectors in V if $\langle u, v\rangle = 0$ for all $u, v \in S$ with $u \neq v$.*

Lemma 6.7. *Let $\{u_1, \ldots, u_n\}$ be an orthogonal set of non-zero vectors in an inner product space V, and suppose $v = \Sigma_{i=1}^n c_i u_i$. Then $c_i = \dfrac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$.*

Proof We have that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$.
Hence $\langle v, u_i\rangle = \langle c_1u_1 + c_2u_2 + \cdots + c_nu_n, u_i\rangle$
$= c_1\langle u_1, u_i\rangle + c_2\langle u_2, u_i\rangle + \cdots + c_n\langle u_n, u_i\rangle$ since the inner product is linear
$= c_i\langle u_i, u_i\rangle$ since $\langle u_j, u_i\rangle = 0$ whenever $j \neq i$.
Thus $c_i = \dfrac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$.

Lemma 6.7 makes it very simple to express any vector as a combination of vectors in an orthogonal basis (an orthogonal basis is an orthogonal set whose elements are linearly independent and spanning).

Example 6.8. $\mathbf{R^2}$ *has an orthogonal basis* $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\} = \{u_1, u_2\}$.

Suppose $v = \begin{pmatrix}1\\2\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$.

Then $c_1 = \dfrac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} = \dfrac{3}{2}$ *and* $c_2 = \dfrac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} = -\dfrac{1}{2}$

i.e., $\begin{pmatrix}1\\2\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}$.

More generally, the theorem tells how to compute coefficients if an expansion of a particular element in V in terms of a particular orthogonal set (which needn't be a basis) exists.

Example 6.9. *Let V be the function space of Example 6.5 and suppose that $\{f_1, f_2, \cdots, f_N\}$ is an orthogonal set in V (it certainly won't be a basis of V if N is finite). Then supposing that we can write*

$$u = c_1f_1 + c_2f_2 + \cdots c_Nf_N$$

it then follows that

$$c_i = \frac{\langle u, f_i\rangle}{\langle f_i, f_i\rangle} = \frac{\int_a^b u(x)f_i(x)\,dx}{\int_a^b f_i^2(x)\,dx}.$$

6.3 Eigenvalues, eigenfunctions and eigenfunction expansions

Let us now return to the Sturm-Liouville problem,

$$-(p(x)y')' + q(x)y = \lambda y, \quad B_a[y] = B_b[y] = 0. \tag{6.48}$$

We shall see that **non-zero** solutions to this problem exist for particular values of λ which we call **eigenvalues** (in analogy to the eigenvalues associated with a finite-dimensional matrix).

Example 6.10. *Find all values of* $\lambda \in \mathbb{R}$ *for which*

$$-y''(x) = \lambda y(x), \quad y(0) = 0 = y(L) \tag{6.49}$$

has non-zero solutions.

Solution: First note that (6.49) has the solution $y(x) = 0$ - we are interested in other possible solutions. Let us examine the three possibilities $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ separately.

<u>Suppose $\lambda < 0$</u>
Then we can let $\lambda = -k^2$ for some $k > 0$ and the equation becomes

$$y'' = k^2 y$$

which has the general solution $y(x) = A\cosh(kx) + B\sinh(kx)$. The two boundary conditions impose the further conditions

$$\begin{aligned} y(0) &= 0 \Leftrightarrow A = 0 \\ y(L) &= 0 \Leftrightarrow B = 0. \end{aligned}$$

The two equations imply that $A = B = 0$. Hence, $y(x) = 0$ is the only solution of (6.49) when $\lambda < 0$. [Alternative solution: The general solution can be written in the form $y(x) = Ae^{kx} + Be^{-kx}$. The two boundary conditions impose the further conditions

$$\begin{aligned} y(0) &= 0 \Leftrightarrow A + B = 0 \quad \implies A = -B \\ y(L) &= 0 \Leftrightarrow Ae^{kL} + Be^{-kL} = 0 \quad \implies A = -Be^{-2kL} \end{aligned}$$

The two equations imply that $A = B = 0$. Hence, $y(x) = 0$ is the only solution of (6.49) when $\lambda < 0$.]

<u>Suppose $\lambda = 0$</u>
The equation is then $y'' = 0$ which has the general solution $y(x) = Ax + B$.

$$\begin{aligned} y(0) &= 0 \Leftrightarrow B = 0 \\ y(L) &= 0 \Leftrightarrow AL = 0. \end{aligned}$$

The two equations imply that $A = B = 0$. Hence, $y(x) = 0$ is the only solution of (6.49) when $\lambda = 0$.

<u>Suppose $\lambda > 0$</u>

Then we can let $\lambda = k^2$ for some $k > 0$ and the equation becomes

$$y'' = -k^2 y$$

which has the general solution $y(x) = A\cos(kx) + B\sin(kx)$. The two boundary conditions impose the further conditions

$$\begin{aligned} y(0) &= 0 \Leftrightarrow A = 0 \\ y(L) &= 0 \Leftrightarrow B\sin(kL) = 0 \Leftrightarrow B = 0 \text{ or } \sin(kL) = 0 \Leftrightarrow B = 0 \text{ or } kL = n\pi, n \in \{1, 2, \cdots\}. \end{aligned}$$

Hence, we find that (6.49) has non-zero solutions when $k = \dfrac{n\pi}{L}$, i.e., $\lambda = \dfrac{n\pi}{L^2}$ for $n \in \{1, 2, \cdots\}$. These non-zero solutions are

$$\sin\left(\frac{\pi x}{L}\right), \ \sin\left(\frac{2\pi x}{L}\right), \ \sin\left(\frac{3\pi x}{L}\right), \cdots .$$

Definition 6.11. *The values of λ for which non-zero solutions of (6.48) exist are called* **eigenvalues**, *the corresponding non-zero solutions are called* **eigenfunctions**.

One of the key properties of eigenfunctions of (6.48) is that they are orthogonal. We shall prove this result, but first a simple Lemma. Suppose we write the Sturm-Liouville problem in the form

$$\begin{aligned} &L[y] = \lambda y, \quad B_a[y] = 0 = B_b[y], \\ &L[y] := -(py')' + qy, \ \ B_a[y] = \alpha_1 y(a) + \beta_1 y'(a), \ \ B_b[y] = \alpha_2 y(b) + \beta_2 y'(b). \end{aligned} \tag{6.50}$$

then we have the following:

Lemma 6.12 (Lagrange's identity)**.** *Let u and v be functions with continuous 2nd derivatives on the interval $[a, b]$ that satisfy the boundary conditions $B_a[y] = 0 = B_b[y]$. Then*

$$\langle L(u), v\rangle = \langle u, L(v)\rangle, \quad \textit{where } \langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Proof Using integration by parts, we have

$$\begin{aligned} -\int_a^b (p(x)u'(x))'v(x)\,dx &= -p(x)u'(x)v(x)|_a^b + \int_a^b p(x)u'(x)v'(x) \\ &= -p(x)u'(x)v(x)|_a^b + p(x)v'(x)u(x)|_a^b - \int_a^b (p(x)v'(x))'u(x)\,dx \\ &= p(x)\left(v'(x)u(x) - u'(x)v(x)\right)|_a^b - \int_a^b (p(x)v'(x))'u(x)\,dx \\ &= -\int_a^b (p(x)v'(x))'u(x)\,dx. \end{aligned}$$

The last step follows after writing $v'(b) = -\dfrac{\alpha_2}{\beta_2}v(b)$, $u'(b) = -\dfrac{\alpha_2}{\beta_2}u(b)$ and similar expressions for $v'(a)$ and $u'(a)$. The four boundary terms cancel. The lemma then follows.

Remark: Since $\langle L(u), v\rangle = \langle u, L(v)\rangle$, the Sturm-Liouville problem (6.50) is called a **symmetric problem** (or sometimes **self-adjoint**, although this term has a wider meaning).

Lemma 6.13. *Suppose that u and v are eigenfunctions of (6.50) corresponding to distinct eigenvalues λ and μ. Then u and v are orthogonal, i.e.,*

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,dx = 0.$$

Proof: From Lagrange's identity, We have

$$\lambda\langle u, v\rangle = \langle L(u), v\rangle = \langle u, L(v)\rangle = \langle u, v\rangle\mu.$$

Hence, $(\lambda - \mu)\langle u, v\rangle = 0$ and so $\langle u, v\rangle = 0$.

Many more results can be proved about the eigenfunctions and eigenvalues of (6.48). The key additional ones (which we state without proof) are:

(i) The eigenvalues are simple; that is to each eigenvalue there corresponds one linearly independent eigenfunction.

(ii) The eigenvalues form an infinite sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

(iii) Let $\{\phi_n\}$, $n \in \{1, 2, \cdots\}$, denote the corresponding set of eigenfunctions (which we know to be orthogonal from the above Lemma). Let f and f' be piecewise continuous functions $[a, b] \to \mathbb{R}$. Then the series

$$\sum_{n=1}^{\infty} c_n\phi_n(x), \quad c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} \tag{6.51}$$

converges to $(f(x_+) + f(x_-))/2$ for all $x \in (a, b)$ (in particular, the series will converge to the value $f(x)$ if the function is continuous at $x \in (a, b)$).

(6.51) is know as an **eigenfunction expansion** of the function f.

Example 6.14. $-y'' = \lambda y, \quad y(0) = 0 = y(L)$.

This is the example we considered above. We have
Eigenvalues: $\frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \cdots$
Eigenfunctions: $\sin(\frac{\pi x}{L}), \sin(\frac{2\pi x}{L}), \sin(\frac{3\pi x}{L}), \cdots$

Eigenfunction expansion: The eigenfunction expansion of a function $f : [0, L] \to \mathbb{R}$ is

$$\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}), \quad c_n = \frac{\langle f, \sin(n\pi x/L)\rangle}{\langle \sin(n\pi x/L), \sin(n\pi x/L)\rangle} = \frac{\int_0^L f(x)\sin(n\pi x/L)\,dx}{\int_0^L \sin^2(n\pi x/L)\,dx}.$$

Let us compute the denominator of c_n. We have

$$\int_0^L \sin^2(n\pi x/L)\,dx = \frac{1}{2}\int_0^L (1-\cos(2n\pi x/L))\,dx = \frac{L}{2},$$

and hence the eigenfunction expansion of $f(x)$ is

$$\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}), \quad c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx.$$

This particular eigenfunction expansion of $f(x) : [0, L] \to \mathbb{R}$ is called the **Fourier sine series**. You will consider such Fourier series in detail in other courses - here they appear as examples of eigenfunction expansions associated with the boundary value problem of Example 6.14.

Example 6.15. *Find the Fourier sine series of the function $f(x) = x$ on the interval* $[0, 1]$.

Solution: We have

$$\begin{aligned} c_n = 2\int_0^1 x\sin(n\pi x)\,dx &= -\frac{2x}{n\pi}\cos(n\pi x)|_0^1 + \int_0^1 \frac{2}{n\pi}\cos(n\pi x)\,dx \\ &= \frac{2(-1)^{n+1}}{n\pi}. \end{aligned}$$

Thus the Fourier sine series of $f(x) = x$ is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x) = \frac{2}{\pi}\left(\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \cdots\right).$$

Since, $f(x) = x$ is continuous, we know that this series should converge to $f(x)$ for all $x \in (0, 1)$ and this is indeed the case. Taking $x = \frac{1}{2}$, we can equate

$$\frac{1}{2} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

which gives us the rather nice series expansion of π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Note however, that the series converges to 0 at $x = 1 \notin (0, 1)$.

Example 6.16. *Find the eigenvalues and eigenfunctions of the boundary value problem*

$$-y''(x) = \lambda y(x), \quad y'(0) = 0 = y'(L) \tag{6.52}$$

and find the associated eigenfunction expansion of a function $g : [0, L] \to \mathbb{R}$, which is piecewise continuous and for which g' is piecewise continuous.

Solution: To find the eigenvalues, let us exam the three possibilities $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ separately.

Suppose $\lambda < 0$

Then we can let $\lambda = -k^2$ for some $k > 0$ and the equation becomes

$$y'' = k^2 y$$

which has the general solution $y(x) = A\cosh(kx) + B\sinh(kx)$. The two boundary conditions impose the further conditions

$$\begin{aligned} y'(0) &= 0 \Leftrightarrow B = 0 \\ y'(L) &= 0 \Leftrightarrow A = 0, \quad \text{since } \; k\sinh(kL) \neq 0 \; \text{ for } \; k > 0. \end{aligned}$$

The two equations imply that $A = B = 0$. Hence, $y(x) = 0$ is the only solution of (6.49) when $\lambda < 0$, and there are no eigenvalues $\lambda < 0$.

Suppose $\lambda = 0$

The equation is then $y'' = 0$ which has the general solution $y(x) = Ax + B$.

$$\begin{aligned} y'(0) &= 0 \Leftrightarrow A = 0 \\ y'(L) &= 0 \Leftrightarrow A = 0. \end{aligned}$$

The two equations clearly imply that $A = 0$. However, B is arbitrary, and so $\lambda = 0$ is an eigenvalue with corresponding eigenvector $y(x) = 1$ (since the differential equation and boundary conditions are both linear, the eigenfunctions are only defined up to an overall multiplicative constant; we choose $y(x) = 1$ for simplicity).

Suppose $\lambda > 0$

Then we can let $\lambda = k^2$ for some $k > 0$ and the equation becomes

$$y'' = -k^2 y$$

which has the general solution $y(x) = A\cos(kx) + B\sin(kx)$. The two boundary conditions impose the further conditions

$$\begin{aligned} y'(0) &= 0 \Leftrightarrow B = 0 \\ y'(L) &= 0 \Leftrightarrow -Ak\sin(kL) = 0 \Leftrightarrow A = 0 \; \text{ or } \; \sin(kL) = 0 \Leftrightarrow A = 0 \; \text{ or } \; kL = n\pi, n \in \{1, 2, \cdots\}. \end{aligned}$$

Hence, we find that (6.52) has eigenvalues

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad n \in \{0, 1, 2, \cdots\},$$

and corresponding eigenfunctions

$$\cos\left(\frac{n\pi x}{L}\right), \quad n \in \{0, 1, 2, \cdots\}$$

The eigenfunction expansion of $g(x)$ is then

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L), \tag{6.53}$$

with

$$\begin{aligned} a_0 &= \frac{\langle g, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{L}\int_0^L g(x)\,dx, \\ a_{n>1} &= \frac{\langle g, \cos(n\pi x/L)\rangle}{\langle \cos(n\pi x/L), \cos(n\pi x/L)\rangle} = \frac{2}{L}\int_0^L g(x)\cos(n\pi x/L)\,dx. \end{aligned}$$

Here, we have used the integral

$$\int_0^L \cos^2(n\pi x/L)\,dx = \frac{1}{2}\int_0^L (1 + \cos(2n\pi x/L)\,dx = \frac{L}{2}.$$

(6.53) is know as the **Fourier cosine series** expansion of $g(x) : [0, L] \to \mathbb{R}$.

Example 6.17. *Determine all the eigenvalues and eigenfunctions of*

$$-y''(x) = \lambda y(x), \quad y(0) = 0, \ \ y(1) + y'(1) = 0 \tag{6.54}$$

and find the associated eigenfunction expansion of a function $f(x)$.

Solution: To find the eigenvalues, let us exam the three possibilities $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ separately.

Suppose $\lambda < 0$

Then we can let $\lambda = -k^2$ for some $k > 0$ and the equation becomes

$$y'' = k^2 y$$

which has the general solution $y(x) = A\cosh(kx) + B\sinh(kx)$. The two boundary conditions impose the further conditions

$$\begin{aligned} y(0) &= 0 \Leftrightarrow A = 0 \\ y(1) + y'(1) &= 0 \Leftrightarrow B(\sinh(k) + k\cosh(k)) = 0 \Leftrightarrow B = 0 \text{ or } -k = \tanh(k). \end{aligned}$$

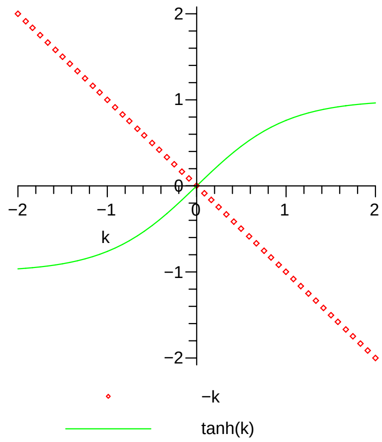
Thus we will only obtain an eigenvalue if there is a solution $k > 0$ of the equation $-k = \tanh(k)$. It is sufficient to sketch the graphs of the two functions $-k$ and $\tanh(k)$ and see if there are any intersection points. Clearly, from Figure 6.3, there are no intersection points $k > 0$ and hence no eigenvalues $\lambda < 0$.

Suppose $\lambda = 0$

The equation is then $y'' = 0$ which has the general solution $y(x) = Ax + B$.

$$\begin{aligned} y(0) &= 0 \Leftrightarrow B = 0 \\ y(1) + y'(1) &= 0 \Leftrightarrow 2A = 0. \end{aligned}$$

Figure 6.3: The graphs of $-k$ and $\tanh(k)$



The two equations clearly imply that $A = B = 0$. Hence, there is no $\lambda = 0$ eigenvalue.

Suppose $\lambda > 0$
Then we can let $\lambda = k^2$ for some $k > 0$ and the equation becomes

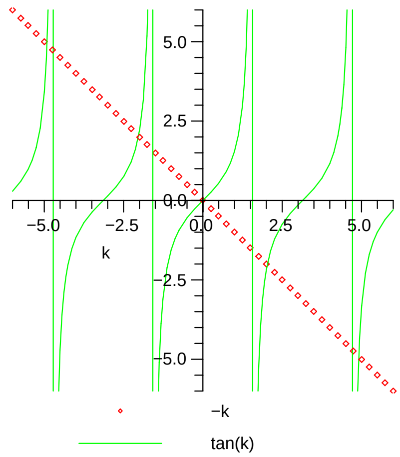
$$y'' = -k^2 y$$

which has the general solution $y(x) = A\cos(kx) + B\sin(kx)$. The two boundary conditions impose the further conditions

$$\begin{aligned} y(0) &= 0 \Leftrightarrow A = 0 \\ y(1) + y'(1) &= 0 \Leftrightarrow B(\sin(k) + k\cos(k)) = 0 \Leftrightarrow B = 0 \text{ or } -k = \tan(k). \end{aligned}$$

Thus we will only obtain an eigenvalue if there is a solution $k > 0$ of the equation $-k = \tan(k)$. Again, we can sketch the graphs of the two functions $-k$ and $\tan(k)$ and see if there are any intersection points. Clearly, from Figure 6.4, there are an

Figure 6.4: The graphs of $-k$ and $\tan(k)$



infinite number of intersection points $0 < k_1 < k_2 < k_3 < \cdots$, with $\pi/2 < k_1 < 3\pi/2$, $3\pi/2 < k_2 < 5\pi/2$ etc. Thus we obtain an infinite number of eigenvalues

$$\lambda = k_n^2, \quad n \in \{1, 2, 3, \cdots\},$$

with corresponding eigenfunctions $\sin(k_n x)$.

A function $f(x) : [0, 1] \to \mathbb{R}$ that is piecewise continuous with piecewise continuous derivative $f'(x)$ has an eigenfunction expansion

$$\sum_{n=1}^{\infty} c_n \sin(k_n x), \quad c_n = \frac{\langle f(x), \sin(k_n x)\rangle}{\langle \sin(k_n x), \sin(k_n x)\rangle}.$$

The denominator of c_n is given by

$$\begin{aligned}\int_0^1 \sin^2(k_n x)\, dx &= \frac{1}{2}\int_0^1 (1 - \cos(2k_n x))\, dx = \frac{1}{2} - \frac{1}{4k_n}\sin(2k_n x)\Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{4k_n}\sin(2k_n x) = \frac{1}{2}\left(1 - \frac{1}{k_n}\sin(k_n x)\cos(k_n x)\right) \\ &= \frac{1}{2}\left(1 + \cos^2(k_n x)\right).\end{aligned}$$

In the last step, we have used the fact that $-k_n = \tan(k_n)$. Thus we arrive at the following eigenfunction expansion for $f(x)$:

$$\sum_{n=1}^{\infty} c_n \sin(k_n x), \quad c_n = \frac{2}{1 + \cos^2(k_n x)} \int_0^1 f(x) \sin(k_n x)\, dx.$$