7 Phase Planes

7.1 Introduction

As we have discussed in detail in Section 2, a system of N ODEs can be expressed as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(t, \boldsymbol{x}(t))$$

where

$$\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^N, \quad \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} \tag{7.1}$$

$$\boldsymbol{F} : \mathbb{R}^{N+1} \to \mathbb{R}^N, \quad \boldsymbol{F}(t, \boldsymbol{x}) = \begin{pmatrix} F_1(t, x_1, \cdots, x_N) \\ \vdots \\ F_N(t, x_1, \cdots, x_N) \end{pmatrix} \tag{7.2}$$

A system is called **autonomous** if it is of the form

$$\dot{\boldsymbol{x}}(t) = \mathbf{F}(\boldsymbol{x}(t)) \tag{7.3}$$

i.e., if there is no explicit t dependence on the right hand side.

Example 7.1. *(i)* $\dot{x}(t) = t\, x(t)$ *is not autonomous.*

(ii) $\dot{x}(t) = x^2(t)$ *is autonomous.*

(iii) The equation $\ddot{x}(t) + \dot{x}(t) + \sin(x(t)) = 0$ *can be expressed as the autonomous system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x_2}(t) &= -\sin(x_1(t)) - x_2(t). \end{aligned}$$

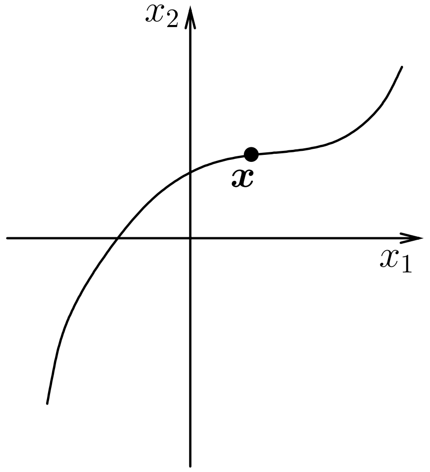
where $x(t) = x_1(t)$.

If we assume that $\mathbf{F}$ is sufficiently smooth, then Picard's theorem tells us that the solution to the initial value problem

$$\dot{\boldsymbol{x}}(t) = \mathbf{F}(\boldsymbol{x}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{7.4}$$

is unique.

Definition 7.2. *The set of points in* $\boldsymbol{x}(t) \in \mathbb{R}^N$ *satisfying (7.4) is called the* **trajectory** *for (7.4) passing through* $\boldsymbol{x}_0$.



Physical Interpretation (N=2)

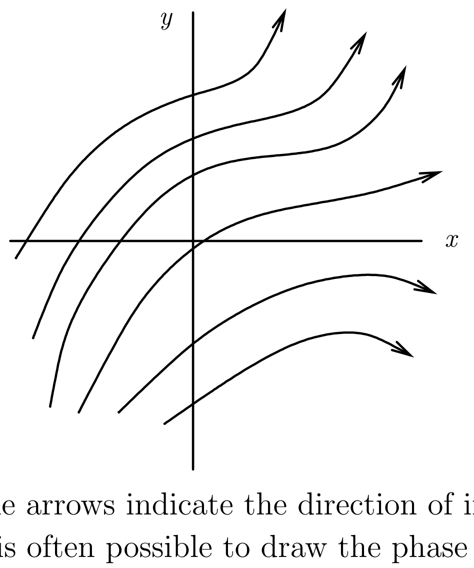
When $N = 2$, we tend to use the notation (x, y) instead of (x_1, x_2). So we have a systems of autonomous equations of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), y(t)) \\ \dot{y}(t) &= g(x(t), y(t)) \end{aligned} \tag{7.5}$$

An interpretation of these equations is that they specify the two components of the velocity of a particle moving in the $x - y$ plane. The particle's velocity at the point (x, y) is given by $(f(x, y), g(x, y))$.

Definition 7.3. *The* **phase plane** *of (7.3) consists of* $\mathbb{R}^N$ *with the trajectories of (7.3) drawn through each point.*

Example 7.4. *The phase plane of (7.5) shows all possible paths which can be followed by the particle for different starting positions. This might be of the form*



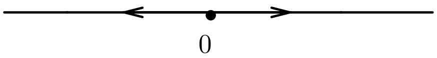
The arrows indicate the direction of increasing t.
It is often possible to draw the phase plane for autonomous equations without solving the equations completely, and then deduce the qualitative nature of of the solutions from the phase plane.

Definition 7.5. *$\boldsymbol{x}_0 \in \mathbb{R}^N$ is called an* **equilibrium point** *of (7.3) if $\boldsymbol{F}(\boldsymbol{x}_0) = 0$.*

If $\boldsymbol{x}_0$ is an equilibrium point of (7.3), $\boldsymbol{x}(t) \equiv \boldsymbol{x}_0$ is a constant solution of (7.3). Hence the trajectory of (7.3) through $\boldsymbol{x}_0$ is equal to $\{\boldsymbol{x}_0\}$; that is, the trajectory is the single point $\boldsymbol{x}_0$.
The first step in finding the phase plane is to find all the equilibrium points.

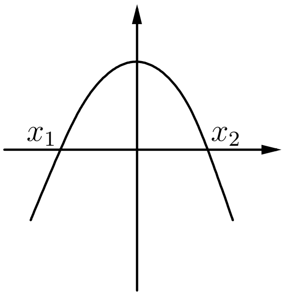
Example 7.6. *Find the phase plane of the equation $\dot{x}(t) = x(t)$ (corresponding to the case $(N = 1,\ \ f(x) = x)$*

Solution: First, we note that 0 is the only equilibrium point. Then it follows that if $x > 0$ we have $\dot{x} > 0$. Similarly if $x < 0$ we have $\dot{x} < 0$. The phase plane consists of the set of possible trajectories in $\mathbb{R}$ and is of the form



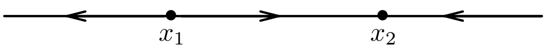
Note that in this simple case we can write the solution exactly $x(t) = x(0)\mathrm{e}^t$.

Example 7.7. *Find the phase plane of the equation $\dot{x}(t) = f(x(t))$ where f has the graph* $(N = 1)$

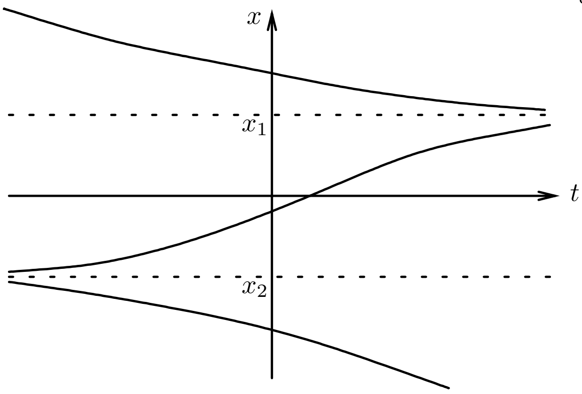


Solution: The equilibrium points are $x = x_1$ and $x = x_2$

If $x < x_1$, $f(x) < 0$ and so $\dot{x} < 0$. If $x_1 < x < x_2$, $f(x) > 0$ and so $\dot{x} > 0$. If $x > x_2$, $f(x) < 0$ and so $\dot{x} < 0$. Hence we have the phase plane



and so we can deduce that solutions have graphs of the form



Example 7.8. *Find the phase plane of* $\ddot{x}(t) = -x(t))$
(This is the equation of the simple harmonic oscillator with general solution $x(t) = A\cos(t) + B\sin(t)$ *- but we are interested in understanding the qualitative nature of the solution.)*

Solution: The equation may be written as the system

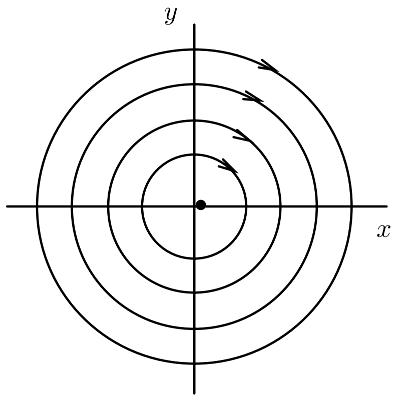
$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned}$$

The only equilibrium point is $(x, y) = (0, 0)$.
We seek trajectories of the form $y = y(x)$.
Then we must have

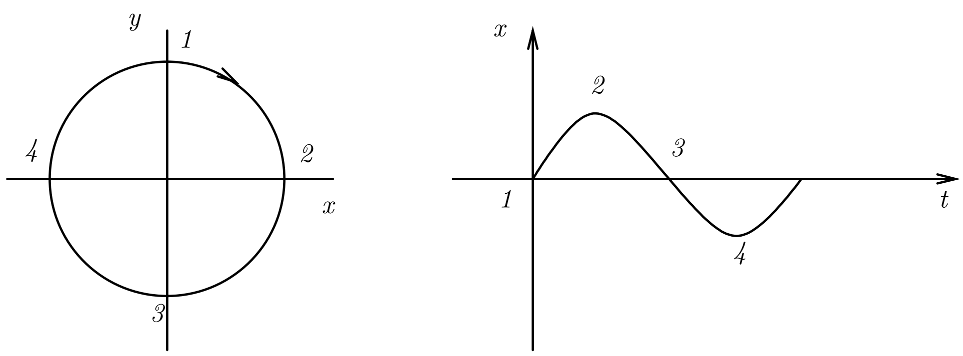
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x}{y}$$

ie $y\,dy = -xdx$ and so $x^2 + y^2 = c$

Hence the trajectories are circles centred on $(0, 0)$ and the phase plane is



The fact that $\frac{dx}{dt} > 0$ when $y > 0$ gives the direction of the arrows in this picture. Since the trajectories are closed they correspond to periodic solutions of the form



Example 7.9. *Find the phase plane of* $\ddot{x}(t) = -x^3(t))$

Solution: The equation may be written as the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x^3$$

Again the only equilibrium point is $(x, y) = (0, 0)$.

Any trajectory of the form $y = y(x)$ must satisfy

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x^3}{y}$$

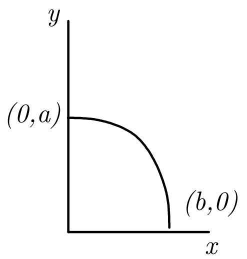
ie $ydy = -x^3dx$ and so $\frac{1}{2}y^2 = -\frac{1}{4}x^4 + c$

i.e. $y^2 = -\frac{1}{2}x^4 + c$

Let us consider the trajectory through the point $(0, a > 0)$. We have

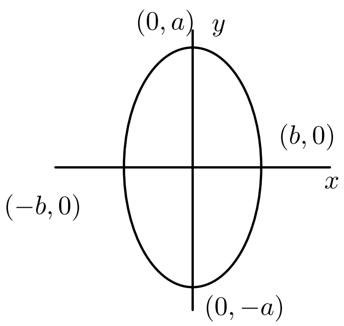
$$y^2 = -\frac{1}{2}x^4 + a^2. \tag{7.6}$$

As we increase x from 0, y^2 and hence y will decrease until it reaches 0 at the point when $x = b := (2a^2)^{1/4}$. The corresponding trajectory in the first quadrant is

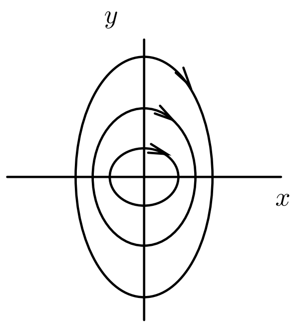


The angles of intersection of the trajectory with the axes can be obtained by considering $\frac{dy}{dx}$ when $x = 0$ and when $y = 0$.

The equation for the trajectory (7.6) is symmetric with respect to the x and y axes (i.e., if (x, y) is a point in the trajectory, then $(-x, y), (x, -y), (-x, -y)$ are also points in the trajectory). Thus we obtain the full trajectory



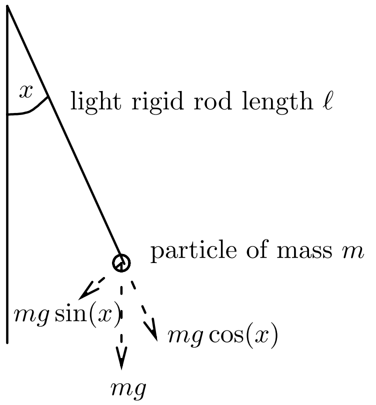
Hence the phase plane is



and all solutions are periodic.

The pendulum equation

Consider the following diagram of forces:



Considering the motion perpendicular to the rod, we have the pendulum equation

$$\begin{aligned} m(\ddot{\ell x}) &= -mg\sin x \quad \text{or} \\ \ddot{x} &= -k^2\sin(x) \quad \text{where } k^2 = \frac{g}{l} \end{aligned}$$

Remark 7.10. *If x is small,* $\sin x \approx x$ *and the equation can be approximated by*

$$\ddot{x} = -k^2 x \tag{7.7}$$

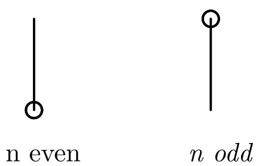
which is the simple harmonic oscillator equation.

Example 7.11. *Find the phase plane of the pendulum equation equation $\ddot{x} = -k^2\sin(x)$.*

Solution: The pendulum equation may be written as the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -k^2\sin x \end{aligned}$$

The equilibrium points are $(x, y) = (n\pi, 0)$ for $n = 0, \pm1, \pm2, \ldots$ corresponding to equilibrium positions



Any trajectory of the form $y = y(x)$ must satisfy

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{k^2 \sin x}{y}$$

ie $y\,dy = -k^2 \sin x dx$ and so

$$y^2 = 2k^2 \cos x + c. \tag{7.8}$$

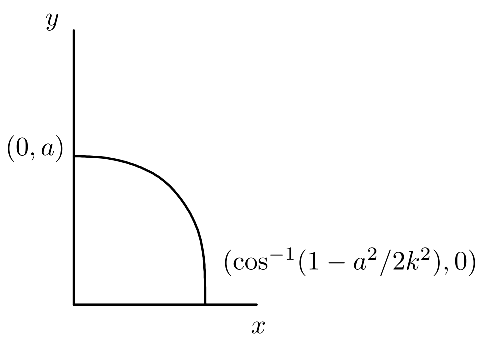
Thus the trajectory passing through $(0, a)$ where $a > 0$ has the equation

$$y^2 = 2k^2(\cos(x) - 1) + a^2 \tag{7.9}$$

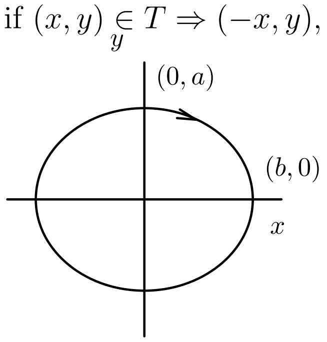
The function $2k^2(\cos(x) - 1)$ decreases from 0 to $-4k^2$ as x increases from 0 to π. The form of the trajectory will thus depend upon whether $a^2 < 4k^2$, $a^2 > 4k^2$ $a^2 = 4k^2$, and we consider theses cases separately.

$a^2 < 4k^2$

As x increases from 0 towards π, y^2 decreases until it reaches $y^2 = 0$ at a value of x such that $\cos x = \frac{2k^2 - a^2}{2k^2} = 1 - \frac{a^2}{2k^2}$.
The corresponding trajectory in the first quadrant is



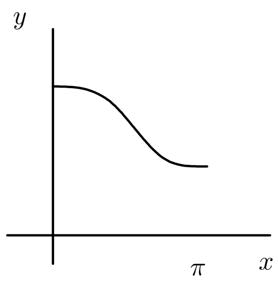
Since equation (7.9) for the trajectory is symmetric with respect to the x and y axes, (ie if $(x, y) \in T \Rightarrow (-x, y), (x, -y), (-x, -y) \in T$)) the full trajectory is



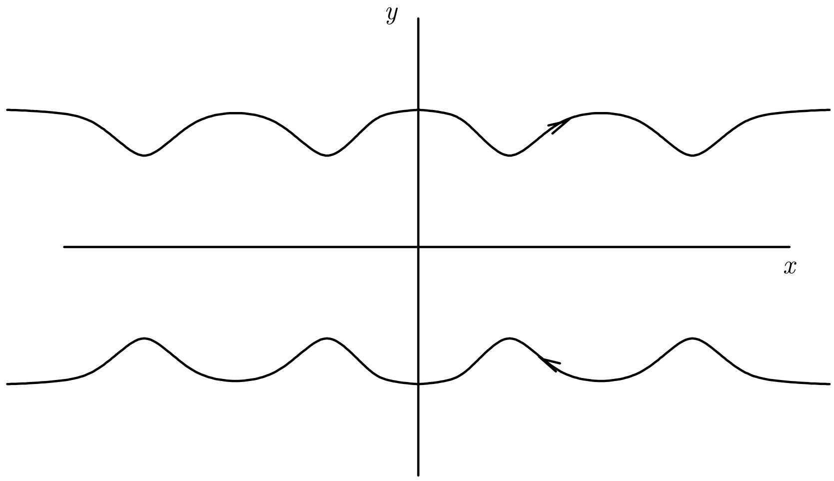
where $b = \cos^{-1}(1 - a^2/2k^2)$. The corresponding solution is periodic - the pendulum oscillates between $x = \pm b$ (like the simple harmonic oscillator).

$a^2 > 4k^2$

As x increases from 0 to π, y^2 decreases fro a^2 to a minimum value of $a^2 - 4k^2$ when $x = \pi$. Hence we obtain the trajectory



The gradient at $x = \pi$ is obtained from $\frac{dy}{dx} = -k^2 \sin(\pi)/\sqrt{a^2 - 4k^2} = 0$. By symmetry in x and y and 2π periodicity in x we obtain

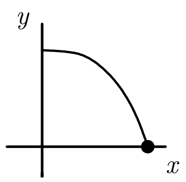


These trajectories correspond to solutions where the pendulum describes complete revolutions. (Every parents nightmare at the swing park!)

$\underline{a^2 = 4k^2}$

Then (7.9) becomes $y^2 = 2k^2(\cos x + 1)$.

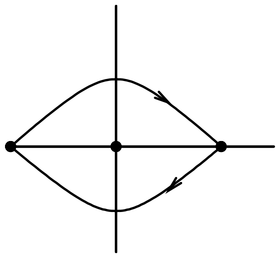
Hence as $x \to \pi$, $y \to 0$, ie the trajectory approaches the equilibrium point $(\pi, 0)$



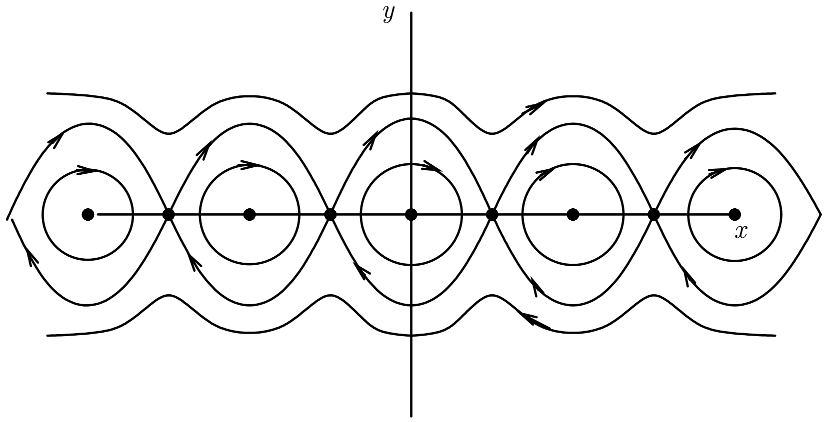
This trajectory corresponds to the pendulum starting from  and approaching

 as $t \to \infty$.

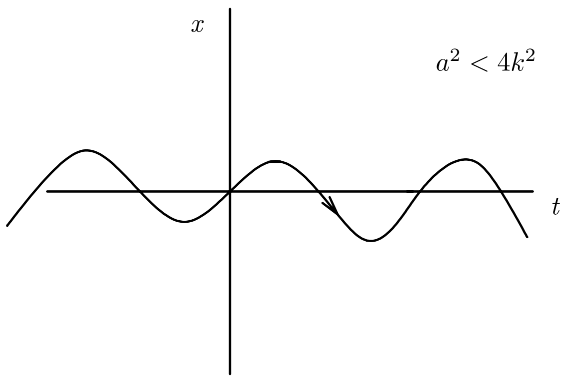
By symmetry we obtain the trajectories

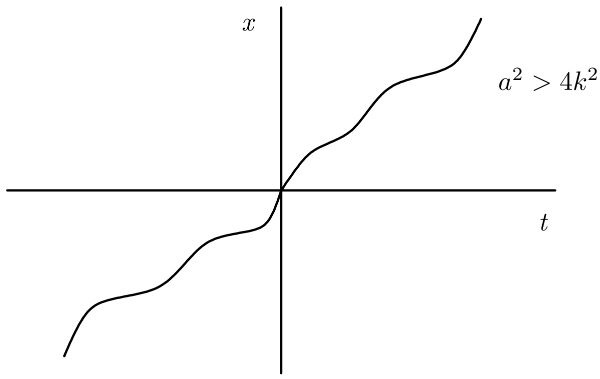


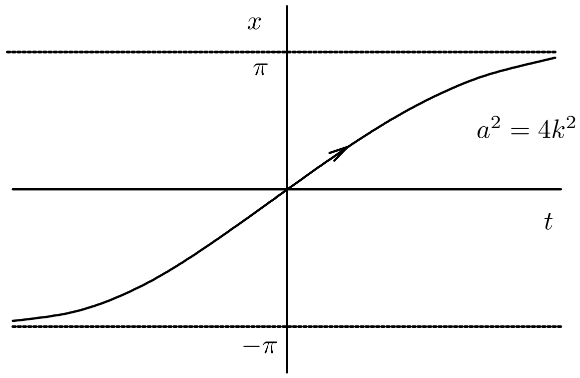
Since all trajectories are periodic in x with period 2π we finally obtain the phase plane



Hence we obtain the following graphs of solutions







7.2 Phase Planes for Linear Systems

7.2.1 Types of Equilibrium Points

Consider

$$\begin{aligned}\dot{x}(t) &= ax(t) + by(t)\\ \dot{y}(t) &= cx(t) + dy(t)\end{aligned}$$

which we can write in the form

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) \tag{7.10}$$

with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We have shown that if λ is an eigenvalue of A with corresponding eigenvector $\boldsymbol{v}$, then $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}$ is a solution of (7.10). Hence we can find the general solution of (7.10) and so draw its phase plane. Clearly, $(0,0)$ is always an equilibrium point of (7.10) and will appear in this phase plane.

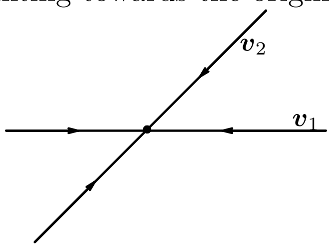
In Section 2, we have discussed the different possibilities that occur for the eigenvalues λ_1, λ_2 of a real matrix A (we still assume A is real). Each of these possibilities will give a different type of phase plane, and we consider them separately.

(a)$\lambda_1 < \lambda_2 < 0$ (i.e. eigenvalues negative and distinct).
The general solution of (7.10) is

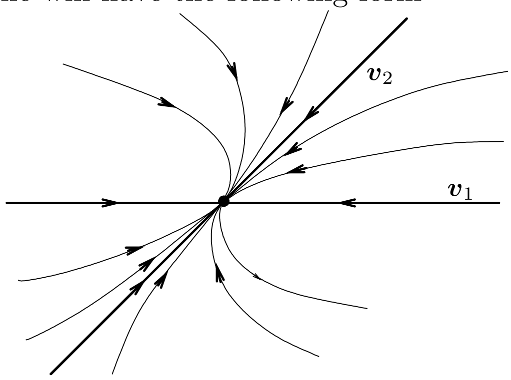
$$\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2. \tag{7.11}$$

First, note that any will solution will $\to (0,0)$ as $t \to \infty$.

Suppose, we have $c_1 > 0, c_2 = 0$. Then the trajectory corresponding to this solution will point along the ray in the direction of $\boldsymbol{v}_1$, with the arrow pointing towards the origin. If $c_1 < 0, c_2 = 0$, the trajectory will point along the ray in the direction of $-\boldsymbol{v}_1$, with the arrow pointing towards the origin. Similarly, if $c_1 = 0, c_2 > 0\,(< 0)$, the trajectory will point along the ray in the direction of $\boldsymbol{v}_2$, $(-\boldsymbol{v}_2)$, with the arrow pointing towards the origin. The situation is

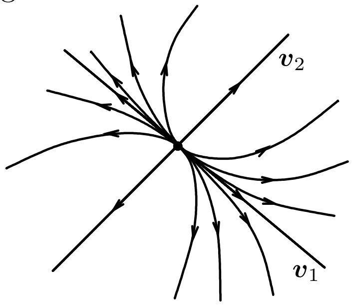


Now consider the trajectory corresponding to the general solution (7.11). Clearly the corresponding trajectory will head into the equilibrium point $(0,0)$ as $t \to \infty$. For $t >> 0$, the $c_2 e^{\lambda_2 t}\boldsymbol{v}_2$ term will dominate and so the direction of approach to $(0,0)$ will be parallel to the direction of $\boldsymbol{v}_2$. Similarly, when $t \to -\infty$ the first term $c_1 e^{\lambda_1 t}\boldsymbol{v}_1$ dominates, and the direction of the trajectory will approach that of $\boldsymbol{v}_1$. The phase plane will have the following form



In this case (0,0) is called a **stable node**.

(b) $0 < \lambda_1 < \lambda_2$ (ie eigenvalues positive and distinct).
The general solution is again given by (7.11). In this case all trajectories will move out from the $(0,0)$ equilibrium point. At $t \to \infty$ the $c_2 e^{\lambda_2 t}\boldsymbol{v}_2$ term will dominate and so the direction of any trajectory will approach that of $\boldsymbol{v}_2$.At $t \to -\infty$ the $c_1 e^{\lambda_1 t}\boldsymbol{v}_1$ term will dominate and so the direction of any trajectory will approach that of $\boldsymbol{v}_1$. The phase diagram will have the form



In this case $(0,0)$ is called an **unstable node**.

Example 7.12. *Sketch the phase plane for the system*

$$\begin{aligned}\dot{x} &= -2x + y \\ \dot{y} &= x - 2y\end{aligned}$$

and hence determine the graphs of x and y as functions of t where x and y are the solutions satisfying $x(0) = 1,\ y(0) = 0$.

Solution: The system may be written as

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

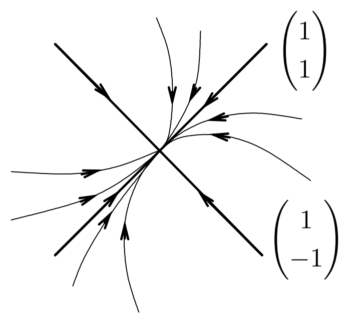
λ is an eigenvalue of A iff $(-2-\lambda)^2 - 1 = 0$ ie $\lambda + 2 = \pm 1$ ie $\lambda = -3,\ -1$.
Hence $(0,0)$ is a stable node.

The eigenvector corresponding to $\lambda = -3$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

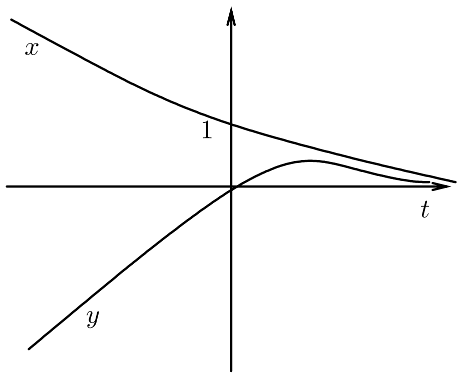
The eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Hence the general solution is $\boldsymbol{x}(t) = c_1 \mathrm{e}^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus the phase plane is



Graphs of solutions satisfying $x(0) = 1, y(0) = 0$ (ie solutions corresponding to trajectory through $(1,0)$) are

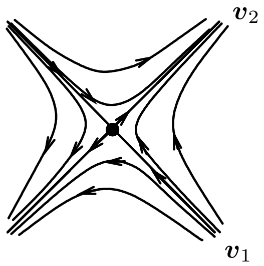


(c) $\lambda_1 < 0 < \lambda_2$ (eigenvalues real and of opposite sign).

The general solution is again given by (7.11).

If $c_1 = 0$, the trajectory is outwards along the ray in the direction of $c_2\boldsymbol{v}_2$. If If $c_2 = 0$, the trajectory is inwards along the ray in the direction of $c_1\boldsymbol{v}_1$.

More generally, when $t \to \infty$ the solution is dominated by $c_2e^{\lambda_2 t}\boldsymbol{v}_2$, and, when $t \to -\infty$ the solution is dominated by $c_1e^{\lambda_1 t}\boldsymbol{v}_1$. Thus the phase plane is of the form



In this case $(0,0)$ is called a **saddle point**.

Example 7.13. *Find the phase plane of*

$$\begin{aligned}\dot{x} &= x + y \\ \dot{y} &= 4x + y\end{aligned}$$

Solution: We may write this system as

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

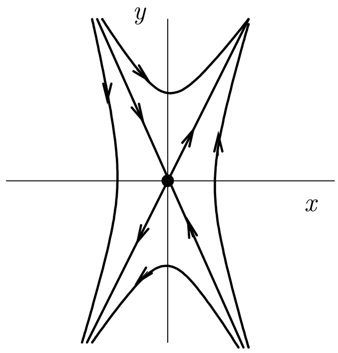
Then λ is an eigenvalue of A iff $(1-\lambda)^2 - 4 = 0$ ie iff $\lambda = -1, 3$.
Hence $(0,0)$ is a saddle point.

Eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Eigenvector corresponding to $\lambda = 3$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus general solution is $\boldsymbol{x}(t) = c_1 \mathrm{e}^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \mathrm{e}^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus the phase plane is

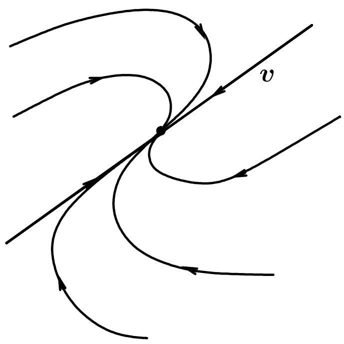


(d) $\lambda_1 = \lambda_2 = \lambda < 0$ (eigenvalues real and equal).

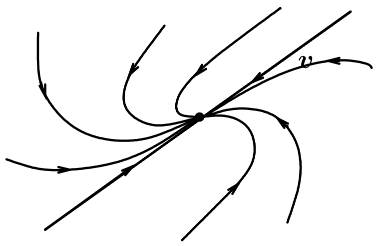
(i) Let us first suppose that there is only one linearly independent eigenvector $\boldsymbol{v}$ corresponding to λ. Then general solution is $\boldsymbol{x}(t) = c_1 \mathrm{e}^{\lambda t}\boldsymbol{v} + c_2 \mathrm{e}^{\lambda t}(t\boldsymbol{v} + \boldsymbol{w})$, where $\boldsymbol{w}$ satisfies $(A - \lambda \mathbb{I})\boldsymbol{w} = \boldsymbol{v}$.

If $c_2 = 0$, we have a solution with a trajectory along the direction of $c_1\boldsymbol{v}$ with direction towards the origin.

The general solution will be dominated by $c_2 e^{\lambda t} t\boldsymbol{v}$ as $t \to \pm\infty$. Hence, we know that all trajectories approach $(0,0)$ as $t \to \infty$ in the direction of $c_2 e^{\lambda t} t\boldsymbol{v}$. In fact there are two possible 'S-shaped' behaviours of the phase plane depending on the relative orientation of $\boldsymbol{v}$ and $\boldsymbol{w}$.



or



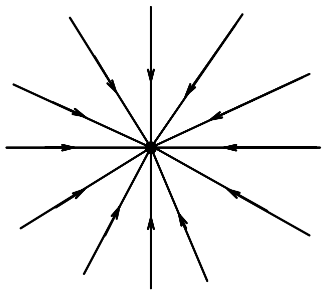
The simplest way to work out which one, is to go back to the original system of equation and consider $\dot{x} = ax + by$. If $b > 0$, then $\dot{x} > 0$ for $x = 0, y >> 0$, and we must have the 1st S-like picture. If $b < 0$, then $\dot{x} < 0$ for $x = 0, y >> 0$, and we must have the 2nd S-like picture.

$(0, 0)$ is called a **stable improper node** in both cases.

(ii) If there are two linearly independent eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ corresponding to λ, then the general solution is

$$y(x) = e^{\lambda t}(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2).$$

Trajectories are in-going rays, and the phase diagram is



which is called a **stable star**.

(e) If $\lambda_1 = \lambda_2 = \lambda > 0$ we obtain a similar phase diagram to the above, but with $|\boldsymbol{x}(t)| \to \infty$ as $t \to \infty$ and $\boldsymbol{x}(t) \to (0, 0)$ as $t \to -\infty$. We find that $(0, 0)$ is an **unstable improper node** or **unstable star** depending on whether there one or two linearly independent eigenvectors.

(f) Complex eigenvalues $\lambda = \alpha + i\beta, \lambda^* = \alpha - i\beta$.
Suppose that the eigenvector corresponding to λ is $\boldsymbol{v} = \boldsymbol{u} + i\boldsymbol{w}$.
The system of ODEs has two linearly independant real solutions obtained by taking the real and imaginary parts of $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}$. $\boldsymbol{x}(t)$ has components

$$\begin{aligned} x(t) &= e^{\lambda t}(u_1 + iw_1) = R_1 e^{\lambda t} e^{i\theta_1} = R_1 e^{\alpha t} e^{i(\beta t + \theta_1)} \\ y(t) &= e^{\lambda t}(u_2 + iw_2) = R_2 e^{\lambda t} e^{i\theta_2} = R_2 e^{\alpha t} e^{i(\beta t + \theta_2)} \end{aligned} \qquad (\dagger)$$

$$\text{where} \quad R_i = \sqrt{u_i^2 + w_i^2}; \;\; \theta_i = \tan^{-1}\left(\frac{w_i}{u_i}\right)$$

So taking real parts, we get the solution

$$\begin{aligned} x(t) &= R_1 e^{\alpha t}\cos(\beta t + \theta_1) \\ y(t) &= R_2 e^{\alpha t}\cos(\beta t + \theta_2) \end{aligned}$$

Note, that these function have the 'quasi' periodicity

$$x(t + \frac{2\pi}{|\beta|}) = e^{\frac{2\pi\alpha}{|\beta|}} x(t)$$
$$y(t + \frac{2\pi}{|\beta|}) = e^{\frac{2\pi\alpha}{|\beta|}} y(t).$$

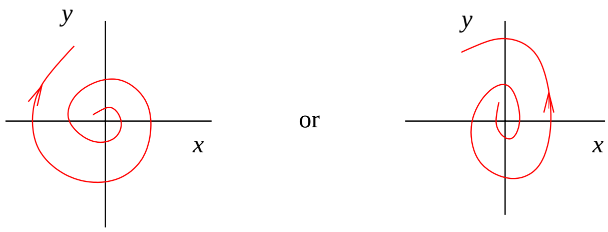
This means that when $t \to t + 2\pi/|\beta|$, the radial direction of a point a trajectory is maintained, but the distance from the origin changes by the factor $e^{2\pi\alpha/|\beta|}$.

Hence we find that that the trajectory is

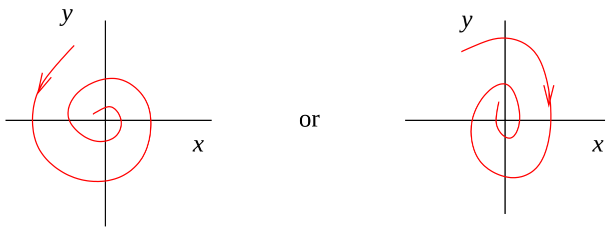
(i) an outwards spiral if $\alpha > 0$.
(ii) an inwards stable point if $\alpha < 0$.
(iii) a periodic orbit if $\alpha = 0$.

For these different cases the equilibrium point $(0, 0)$ is referred as a

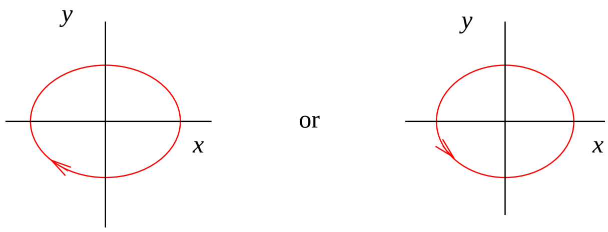
(i) **unstable spiral point** $(\alpha > 0)$



(ii) **stable spiral point** $(\alpha < 0)$



(iii) **centre** $(\alpha = 0)$



To works out the clockwise or anticlockwise orientations of each of these possibilities, we consider $\dot{x}$ at $(x, y) = (0, 1)$.
$\dot{x} = ax + by = b$ at $(0, 1)$. So if $b > 0$, our trajectories will be clockwise, and if $b < 0$ they are anticlockwise.

Remark 7.14. *If we take the imaginary part of $e^{\lambda t}\boldsymbol{v}$ we find the solution*

$$\begin{aligned} x(t) &= R_1 e^{\alpha t}\sin(\beta t + \theta_1) \\ y(t) &= R_2 e^{\alpha t}\sin(\beta t + \theta_2) \end{aligned}$$

This gives the same trajectories. There is nothing new. Either is sufficient to give all trajectories.

Example 7.15. *Find the phase plane of*

$$\begin{aligned} \dot{x} &= 3x - y \\ \dot{y} &= 5x - y \end{aligned}$$

Solution: We can write the system as

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

λ is an eigenvalue of A iff

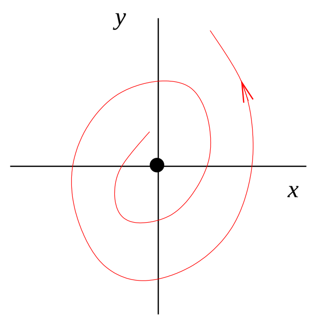
$$\begin{vmatrix} 3-\lambda & -1 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

ie $(\lambda - 3)(\lambda + 1) + 5 = 0$ ie $\lambda^2 - 2\lambda + 2 = 0$ ie $\lambda = \frac{2\pm\sqrt{4-8}}{2} = 1 \pm i$

Hence $(0, 0)$ is an unstable spiral point . Note that direction of spiral is determined by the fact that $\dot{x} < 0$ on the positive y axis (at $(0, 1)$ for example). The spiral can be elongated, and skewed with respect to the axes. A rough idea of the shape may be gained by looking at $\frac{dy}{dx}$ at the intersects with the x and y axis. We have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{5x - y}{3x - y}.$$

Thus on the x axis (corresponding to $y = 0$ and $x \neq 0$) we have $\frac{dy}{dx} = \frac{5}{3}$. On the y axis (corresponding to $x = 0$ and $y \neq 0$) we have $\frac{dy}{dx} = 1$. Thus, the phase plane is of the form



(g) $\lambda_1 = 0$ (ie one eigenvalue = 0)
If $\boldsymbol{v}_1$ is the eigenvector corresponding to λ_1, $A\boldsymbol{v}_1 = 0$. Hence $\boldsymbol{x}(t) = c\boldsymbol{v}_1$ is a solution of $\dot{\boldsymbol{x}} = A\boldsymbol{x}(t)$ for all c.
Hence $c\boldsymbol{v}_1$ is an equilibrium point for all c.
Also, A has $\det(A) = 0$, is not invertible and its rows are multiples of each other.
This latter fact makes it possible to find equations of the trajectories in the form $y = ax + b$.

Example 7.16. *Find the phase plane for the system*

$$\begin{aligned}\dot{x} &= x - y\\ \dot{y} &= 2x - 2y\end{aligned}$$

Solution: Clearly

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

has 0 determinant, and so 0 is an eigenvalue of A.
$(1-\lambda)(-2-\lambda) + 2 = 0 \Rightarrow \lambda = 0, \lambda = -1$.

(x, y) is an equilibrium point iff

$$\begin{aligned}x - y &= 0\\ 2x - 2y &= 0\\ \text{ie iff } x &= y\end{aligned}$$

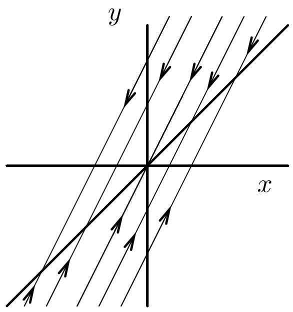
Thus $x = y$ is a line of equilibrium points.

Also $y = y(x)$ is a trajectory for the system iff

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2x - 2y}{x - y} = 2$$

ie iff $y = 2x + c$.

Hence we have the phase plane



The direction of the arrows comes from the observation that $\dot{x} < 0$ when $y > x$ and $\dot{x} > 0$ when $y > x$.

7.2.2 Stability

Consider the autonomous system of equations

$$\dot{\boldsymbol{x}}(t) = \mathbf{F}(\boldsymbol{x}(t)) \tag{7.12}$$

Roughly speaking an equilibrium point $\boldsymbol{x}_0$ of (7.12) is stable if any solution which starts close to $\boldsymbol{x}_0$ at $t = 0$ stays close to $\boldsymbol{x}_0$ for all $t > 0$.

Definition 7.17. *$\boldsymbol{x}_0$ is a* **stable equilibrium point** *of (7.12) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\boldsymbol{x}$ is a solution of (7.12) with $|\boldsymbol{x}(0) - \boldsymbol{x}_0| < \delta$ then $|\boldsymbol{x}(t) - \boldsymbol{x}_0| < \epsilon$ for all $t \geq 0$ ($|\quad|$ denotes distance in $\mathbb{R}^N$).*

Clearly for linear systems

(i) stable nodes, stable improper nodes, stable stars, stable spiral points and centres are stable equilibrium points.

(ii) unstable nodes, unstable improper nodes, unstable stars, unstable spiral points and saddle points are unstable equilibrium points.

(iii) $(0, 0)$ is a stable equilibrium point of $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ iff real parts of all eigenvalues of A are ≤ 0.

If $\boldsymbol{x}_0$ is a stable equilibrium point and $\boldsymbol{x}(0)$ is close to $\boldsymbol{x}_0$ it is not necessarily true that $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_0$, e.g., if $\boldsymbol{x}_0$ is a centre.

Definition 7.18. *$\boldsymbol{x}_0$ is an* **asymptotically stable** *solution of (7.12) if $\boldsymbol{x}_0$ is a stable solution of (7.12) and there exists $\delta > 0$ such that whenever $\boldsymbol{x}$ is a solution of (7.12) with $|\boldsymbol{x}(0) - \boldsymbol{x}_0| < \delta$ then $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_0$.*

Clearly stable nodes, stable improper nodes, stable stars and stable spiral points are asymptotically stable equilibrium points for linear systems. Centres are stable, but not asymptotically stable.

Summary of Stability Properties (det$A \neq 0$)				
	Eigenvalue	Type	Stability	Lin. Thm?
(a)	$\lambda_1 < \lambda_2 < 0$	stable node	Asym. Stable	✓
(b)	$0 < \lambda_1 < \lambda_2$	unstable node	unstable	✓
(c)	$\lambda_1 = \lambda_2 < 0$ $\lambda_1 = \lambda_2 > 0$	stable improper node or stable star unstable improper node or unstable star	Asym. Stable Unstable	× ×
(d)	$\lambda_1 < 0 < \lambda_2$	saddle point	Unstable	✓
(e)	$\lambda_{1,2} = \alpha \pm i\beta$ $\alpha < 0$ $\alpha > 0$	 stable spiral point unstable spiral point	 Asym. Stable Unstable	 ✓ ✓
(f)	$\lambda_{1,2} = \pm i\beta$	centre	Stable	×