8 Nonlinear Systems

8.1 Linearisation

Consider

$$\left.\begin{aligned}\dot{x}(t) &= f(x(t),y(t))\\ \dot{y}(t) &= g(x(t),y(t))\end{aligned}\right\}\qquad (N)$$

where f and g are smooth functions such that $f(0,0)=0=g(0,0)$ i.e. $(0,0)$ is an equilibrium point.

We now obtain a good linear approximation to (N) close to $(0,0)$. Taking a Taylor expansion we have

$$\begin{aligned} f(x,y) &= f(0,0) + x\frac{\partial f}{\partial x}(0,0) + f\frac{\partial f}{\partial y}(0,0)\\ &+ \frac{1}{2!}\left[x^2\frac{\partial^2 f}{\partial x^2}(0,0) + 2xy\frac{\partial^2 f}{\partial x\partial y}(0,0) + y^2\frac{\partial^2 f}{\partial y^2}(0,0)\right]\\ &+ \text{higher powers in } x \text{ and } y.\\ &\approx x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) \text{ if } x \text{ and } y \text{ are small.}\end{aligned}$$

Similarly

$$g(x,y) \approx x\frac{\partial g}{\partial x}(0,0) + y\frac{\partial g}{\partial y}(0,0) \text{ if } x \text{ and } y \text{ are small.}$$

Hence for (x,y) close to $(0,0)$, (N) is closely approximated by the linear system

$$\left.\begin{aligned}\dot{x}(t) &= \tfrac{\partial f}{\partial x}(0,0)x + \tfrac{\partial f}{\partial y}(0,0)y\\ \dot{y}(t) &= \tfrac{\partial g}{\partial x}(0,0)x + \tfrac{\partial g}{\partial y}(0,0)y\end{aligned}\right\}\qquad (L)$$

(L) is called the linearised equation of (N).

It seems likely that solutions of (N) and solutions of (L) should behave similarly, close to $(0,0)$.
In fact the following can be proved

Theorem 8.1. Linearisation Theorem
Let λ and μ be eigenvalues of

$$\begin{pmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0)\\ \frac{\partial g}{\partial x}(0,0) & \frac{\partial g}{\partial y}(0,0)\end{pmatrix}$$

If Re(λ), Re(μ) $\neq 0$ and $\lambda \neq \mu$, then $(0,0)$ is the same type of equilibrium point for both (N) and (L).
(Possible types - stable/unstable node, stable/unstable spiral point, saddle point).

Remarks

(a) If $\mathrm{Re}(\lambda)$ or $\mathrm{Re}(\mu) = 0$ or if $\lambda = \mu$ we cannot use the above theorem.

(b) If $\lambda = \mu$ with $\lambda, \mu < 0$, then $(0,0)$ is a stable improper node or stable star for (L), but it may be either the same for (N) or become a stable spiral point.

If $\lambda = \mu$ with $\lambda, \mu > 0$, then $(0,0)$ is a unstable improper node or unstable star for (L), but may be either the same or an unstable spiral point for (N).

We can't use the theorem to tell us anything about (N) in these cases.

The theorem can be extended, and it can be also be shown that the nature of an equilibrium point (a,b) of the nonlinear system

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t), y(t)) \\ \dot{y}(t) &= g(x(t), y(t)) \end{aligned}\right\} \qquad (N)$$

is the same as the equilibrium point (0,0) of the linear system

$$\left.\begin{aligned} \dot{x}(t) &= \tfrac{\partial f}{\partial x}(a,b)x + \tfrac{\partial f}{\partial y}(a,b)y \\ \\ \dot{y}(t) &= \tfrac{\partial g}{\partial x}(a,b)x + \tfrac{\partial g}{\partial y}(a,b)y \end{aligned}\right\} \qquad (L)$$

except when $\mathrm{Re}(\lambda) = 0$ or $\mathrm{Re}(\mu) = 0$ or $\lambda = \mu$.

(L) is called the linearised equation of (N) at (a,b). The phase plane of (L) close to $(0,0)$ gives a good approximation to the phase plane of (N) close to (a,b).

Example 8.2. *Find the equilibrium points and determine their nature for the system*

$$\left.\begin{aligned} \dot{x}(t) &= 2y + xy \\ \dot{y}(t) &= x + y \end{aligned}\right\} \qquad (N)$$

and hence plot a possible phase plane.

Solution: (x, y) is an equilibrium point iff

$$\begin{aligned} 2y + xy &= 0 \qquad (1) \\ x + y &= 0 \qquad (2) \end{aligned}$$

$$\begin{aligned} (1) &\iff y(2+x) = 0 \iff y = 0 \text{ or } x = -2 \\ (2) &\iff x = -y \end{aligned}$$

hence equilibrium points are $(0, 0)$ and $(-2, 2)$.

If $f(x, y) = 2y + xy$, $\frac{\partial f}{\partial x}(x, y) = y$ and $\frac{\partial f}{\partial y}(x, y) = 2 + x$
If $g(x, y) = x + y$, $\frac{\partial g}{\partial x}(x, y) = 1$ and $\frac{\partial g}{\partial y}(x, y) = 1$.

Hence the linearised equation at $(0, 0)$ is

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

λ is an eigenvalue of A iff $-\lambda(1 - \lambda) - 2 = 0$. i.e. $\lambda^2 - \lambda - 2 = 0$.
i.e. $(\lambda - 2)(\lambda + 1) = 0$, i.e. $\lambda = -1, 2$. The corresponding eigenvectors are $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Hence $(0, 0)$ is a saddlepoint for (L) and so is also a saddlepoint for (N).

The linearised equation at $(-2, 2)$ is

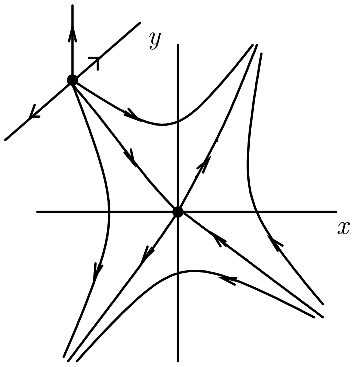
$$\dot{\boldsymbol{x}} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

Since A has eigenvalues $1, 2$, $(0, 0)$ is and unstable node for (L) and so $(-2, 2)$ is an unstable node for (N). Further rough information about the direction of lines can be obtained from the following facts:

$(0, 0)$ has eigenvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalues -1 and 2.

$(-2, 2)$ has eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues 2 and 1.

A possible phase plane for (N) is



Example 8.3. *Find the equilibrium points and determine their nature for the pendulum equation*

$$\ddot{\boldsymbol{x}} = -\sin(\boldsymbol{x}) \qquad\qquad (N)$$

Hence, plot a possible phase plane.

Solution: The corresponding system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) \qquad\qquad (L) \end{aligned}$$

Equilibrium points are $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$

If $f(x, y) = y$, $\frac{\partial f}{\partial x}(x, y) = 0$, $\frac{\partial f}{\partial y}(x, y) = 1$
If $g(x, y) = -\sin(x)$, $\frac{\partial g}{\partial x}(x, y) = -\cos(x)$, $\frac{\partial g}{\partial y}(x, y) = 0$

Hence the linearised equation at $(n\pi, 0)$ is

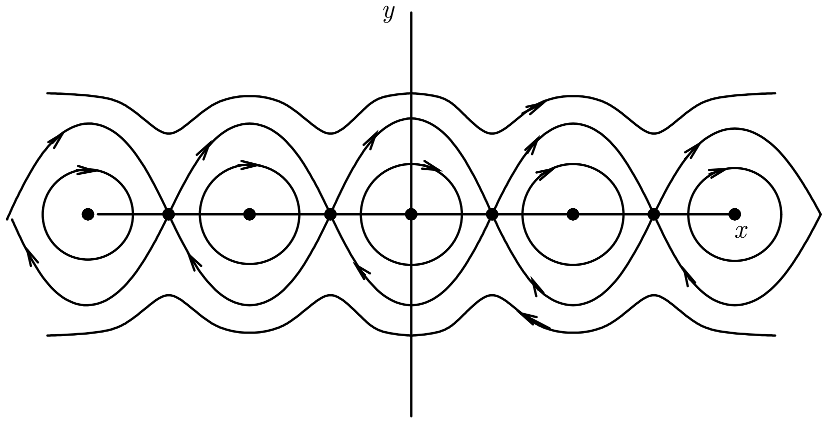
$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\cos(n\pi) & 0 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

The eigenvalues of A satisfy $\lambda^2 = -\cos(n\pi) = (-1)^{n+1}$.

If n is odd, $\lambda^2 = 1$, i.e. $\lambda = -1, 1$ (with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$), and so $(0, 0)$ is a saddle point for (L) and $(n\pi, 0)$ is a saddle point for (N).

If n is even, $\lambda^2 = -1$, i.e. $\lambda = -i, i$. Thus $(0, 0)$ is a centre for (L), but we cannot conclude from the theorem that $(n\pi, 0)$ is a centre for (N).
We showed earlier, however, by considering the equations of trajectories, that $(n\pi, 0)$ is in fact a centre for (N).

The phase plane is



Example 8.4. *If the pendulum is subject to air resistance the equation becomes*

$$\ddot{\boldsymbol{x}} + r\dot{\boldsymbol{x}} + \sin(\boldsymbol{x}) = 0, \qquad (N) \quad (\textit{we assume weak damping with } r^2 < 4)$$

with corresponding system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) - ry \end{aligned}$$

The equilibrium points are again $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, ...$

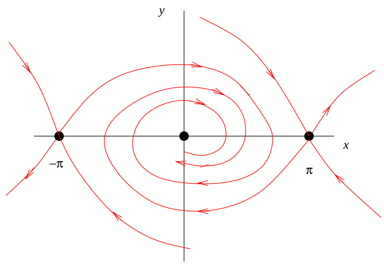
The linearised equation at $(n\pi, 0)$ is

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -\cos(n\pi) & -r \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$$

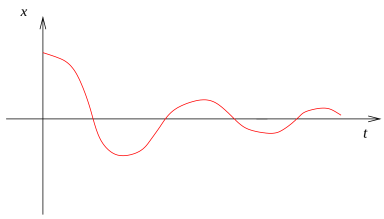
The eigenvalues of A satisfy $\lambda(\lambda + r) + \cos(n\pi) = 0$ i.e. $\lambda^2 + r\lambda + \cos(n\pi) = 0$ i.e. $\lambda = \frac{-r \pm \sqrt{r^2 - 4\cos(n\pi)}}{2}$.

If n is odd, $\lambda = \frac{-r \pm \sqrt{r^2+4}}{2}$, and so $(n\pi, 0)$ is a saddle point for (N) (as in the undamped case).
If n is even, $\lambda = \frac{-r \pm i\sqrt{4-r^2}}{2}$, and so $(n\pi, 0)$ is now a stable spiral point (N).

A <u>possible</u> phase plane is a 2π periodic repetition of



The graph of $\boldsymbol{x}$ as a function of t for trajectory with $(x(0), y(0)) = (1, 0)$ will be



Thus the oscillations die out as we might expect (see Problem Sheet 12 for other cases: $r = 4, r > 4$)

8.2 Competing Species

8.2.1 Introduction

The growth of a single species can be modelled by

$$\dot{x} = ax$$

where a is the **growth rate** of the species. The exact solution is $x(t) = x(0)e^{at}$, and the phase plane is

a<0

a>0

with $x > 0$ being the physically relevant region.

The growth of a species is better modelled by the logistic equation

$$\dot{x} = ax - bx^2 \qquad a > 0, b > 0$$

with phase plane

0 a/b

In this case crowding effects reduces growth rate to $a - bx$, i.e. the population grows if $0 < x < \frac{a}{b}$ and population decreases if $x > \frac{a}{b}$.

Suppose now that two species live in the same region and compete for the same food supply (e.g. red and grey squirrels).
Let x and y denote the population sizes of the two species.
Because of crowding/competition effects the growth rate of x is now given by $a_1 - b_1x - c_1y$ and the growth rate of y is given by $a_2 - b_2y - c_2x$.

Thus x and y satisfy

$$\left.\begin{aligned} \tfrac{dx}{dt} &= x(a_1 - b_1 x - c_1 y) = a_1 x - b_1 x^2 - c_1 xy \\ \\ \tfrac{dy}{dt} &= y(a_2 - b_2 y - c_2 x) = a_2 y - b_2 y^2 - c_2 xy \end{aligned}\right\} \qquad (8.1)$$

Question What happens as $t \to \infty$? Do one/both/neither population die out?

Equilibrium points

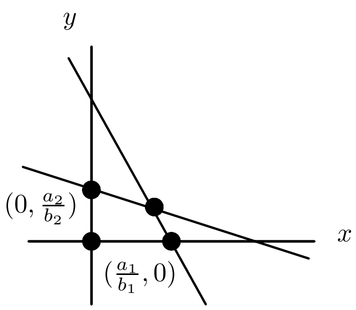
(x, y) is an equilibrium point of (8.1) if

$$\begin{aligned} x(a_1 - b_1 x - c_1 y) &= 0 \\ y(a_2 - b_2 y - c_2 x) &= 0 \end{aligned} \qquad (8.2)$$

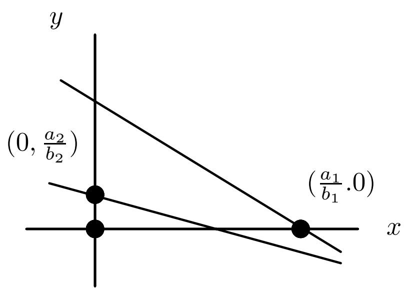
Hence we have equilibrium points $(0,0), (0, \frac{a_2}{b_2}), (\frac{a_1}{b_1}, 0)$ and (x, y) such that

$$\left.\begin{aligned} b_1 x + c_1 y = a_1 \\ c_2 x + b_2 y = a_2 \end{aligned}\right\} \qquad (8.3)$$

Each equation in (8.3) represents a straight line of negative gradient, and so (8.3) may or may not have a solution with $x, y > 0$. (Note the first straight line is through the 2 points $(0, a_1/c_1)$ and $(a_1/b_1, 0)$, and the 2nd is through the points $(0, a_2/b_2)$ and $(a_2/c_2, 0)$). We have



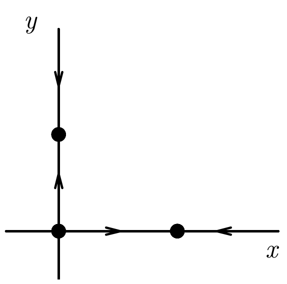
or



Remark

The system *always* has the equilibrium point $(0, 0)$ as well as the equilibrium points $(\frac{a_1}{b_1}, 0)$ and $(0, \frac{a_2}{b_2})$ on the x and y axes.

Note that the x and y axes are themselves trajectories.



Example 8.5. *Find all the equilibrium points for the following system modelling competing species*

$$\frac{dx}{dt} = x(1 - x - y)$$

$$\frac{dy}{dt} = y(\tfrac{1}{2} - \tfrac{1}{4}y - \tfrac{3}{4}x)$$

and determine the nature of each.
Hence sketch a possible phase plane for the system.

Solution: (x, y) is an equilibrium point if

$$x(1 - x - y) = 0$$

$$y(\tfrac{1}{2} - \tfrac{1}{4}y - \tfrac{3}{4}x) = 0$$

Hence the equilibrium points are $(0,0), (0,2), (1,0)$ and (x, y) such that

$$\begin{aligned} x + y &= 1 \\ \tfrac{3}{4}x + \tfrac{1}{4}y &= \tfrac{1}{2} \end{aligned}$$

which has the solution $(\frac{1}{2}, \frac{1}{2})$.

If $f(x, y) = x(1 - x - y)$, $\frac{\partial f}{\partial x} = 1 - 2x - y$, $\frac{\partial f}{\partial y} = -x$
If $g(x, y) = y(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x)$, $\frac{\partial g}{\partial x} = -\frac{3}{4}y$, $\frac{\partial g}{\partial y} = \frac{1}{2} - \frac{1}{2}y - \frac{3}{4}x$

(i) The linearised equation at $(0, 0)$ is

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t)$$

The eigenvalues of A are $\lambda = \frac{1}{2}, 1$ and so $(0, 0)$ is an unstable node.

(ii) The linearised equation at $(0, 2)$ is

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} -1 & 0 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t)$$

The eigenvalues of A are $\lambda = -1, -\frac{1}{2}$ and so $(0, 2)$ is a stable node

(iii) The linearised equation at $(1, 0)$ is

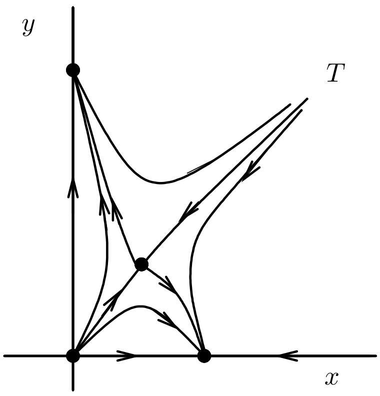
$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t)$$

The eigenvalues of A are $\lambda = -\frac{1}{4}, -1$ and so $(1, 0)$ is a stable node.

(iv) The linearised equation at $(\frac{1}{2}, \frac{1}{2})$ is

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8} \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t)$$

λ is an eigenvalue of A if $(\lambda + \frac{1}{2})(\lambda + \frac{1}{8}) - \frac{3}{16} = 0$
i.e. $\lambda^2 + \frac{5}{8}\lambda - \frac{1}{8} = 0$, i.e. $8\lambda^2 + 5\lambda - 1 = 0$, i.e. $\lambda = \frac{-5\pm\sqrt{57}}{16}$
Hence $(\frac{1}{2}, \frac{1}{2})$ is a saddle point.
A possible phase plane is



Remark
If $\boldsymbol{x}(0)$ lies above T , $\boldsymbol{x}(t) \to (0, 2)$, i.e. x species dies out.
If $\boldsymbol{x}(0)$ lies below T , $\boldsymbol{x}(t) \to (1, 0)$, i.e. y species dies out.
If $\boldsymbol{x}(0)$ lies exactly on T (which is very unlikely), $\boldsymbol{x}(t) \to (\frac{1}{2}, \frac{1}{2})$, and neither species dies out.

8.2.2 Predator- Prey Equations

Now suppose that the x species (say rabbits) is preyed on by the y species (say foxes). Then the presence of y will decrease the growth rate of x (y eats x).
The presence of x will increase the growth rate of y (x is eaten by y).

We also assume that prey population has positive growth rate in absence of predators, and that the predator population has negative growth rate in absence of prey.

$$\begin{aligned} \text{Thus growth rate of prey} &= a - \alpha y \text{ and} \\ \text{growth rate of predators} &= -b + \beta x. \end{aligned}$$

Thus we obtain equations

$$\begin{aligned}\frac{dx}{dt} &= ax - \alpha xy \qquad \text{(prey)}\\ \frac{dy}{dt} &= -by + \beta xy \qquad \text{(predator)}\end{aligned}$$

(x, y) is an equilibrium point if

$$\begin{aligned}x(a - \alpha y) &= 0\\ y(-b + \beta x) &= 0\end{aligned}$$

Hence equilibrium points are $(0, 0)$ and $(\frac{b}{\beta}, \frac{a}{\alpha})$.

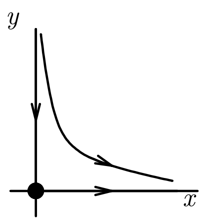
If $f(x, y) = x(a - \alpha y)$, $\frac{\partial f}{\partial x} = a - \alpha y$, $\frac{\partial f}{\partial y} = -\alpha x$

If $g(x, y) = y(-b + \beta x)$, $\frac{\partial g}{\partial x} = \beta y$, $\frac{\partial g}{\partial y} = -b + \beta x$

The linearised equation at $(0, 0)$ is

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t)$$

Eigenvalues of A are $\lambda = -b, a$ and so $(0, 0)$ is a saddle point.



The linearised equation at $(\frac{b}{\beta}, \frac{a}{\alpha})$ is

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 0 & -\frac{\alpha b}{\beta} \\ \frac{\beta a}{\alpha} & 0 \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t)$$

λ is an eigenvalue of A if $\lambda^2 + ab = 0$, i.e. $\lambda = \pm i\sqrt{ab}$.
Thus the linearised equation has a centre at $(0, 0)$, but we can draw no firm conclusion about the nature of the equilibrium point $(\frac{b}{\beta}, \frac{a}{\alpha})$ of the original equation.

However, we can find the equations of the trajectories in the phase plane explicitly.

We have $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y(-b + \beta x)}{x(a - \alpha y)}$

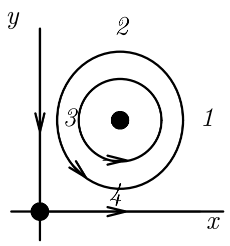
Thus $\int \frac{a-\alpha y}{y}\, dy = \int \frac{-b+\beta x}{x}\, dx$

Integrating, we find that $-a\ln y - b\ln x + \alpha y + \beta x = constant$

The function $(x, y) \mapsto -a\ln y - b\ln x + \alpha y + \beta x$ has a minimum at $(x, y) = (\frac{b}{\beta}, \frac{a}{\alpha})$ (check this by the 2nd derivative test).

The trajectories $-a\ln y - b\ln x + \alpha y + \beta x = constant$ are level curves for the function - they are closed curves surrounding $(\frac{b}{\beta}, \frac{a}{\alpha})$.

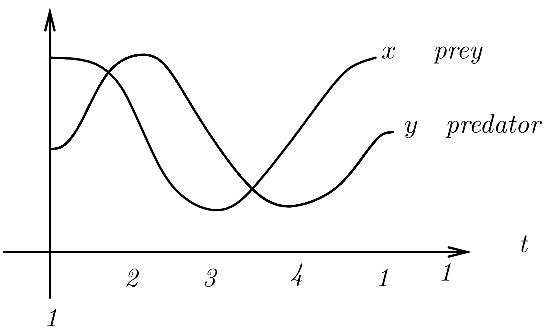
Hence the phase plane is of the form



Remarks

(a) According to this model populations vary in a periodic manner.

(b) There is a difference of $\frac{1}{4}$ period between the behaviour of the predator and prey populations.



i.e. prey achieves maximum at (1) , predator at (2).
prey achieves minimum at (3) , predator at (4).

8.3 Lyapunov Functions

Consider the non-linear system

$$\left.\begin{aligned}\dot{x}(t) &= f(x(t), y(t))\\ \dot{y}(t) &= g(x(t), y(t))\end{aligned}\right\} \qquad (N)$$

with equilibrium point $\boldsymbol{x}_0$.

If the linearised equation at $\boldsymbol{x}_0$, i.e.

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} \frac{\partial f}{\partial x}(\boldsymbol{x}_0) & \frac{\partial f}{\partial y}(\boldsymbol{x}_0) \\ \frac{\partial g}{\partial x}(\boldsymbol{x}_0) & \frac{\partial g}{\partial y}(\boldsymbol{x}_0) \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t) \qquad (L)$$

is such that an eigenvalue for A has zero real part, or eigenvalues are equal, we cannot conclude that $\boldsymbol{x}_0$ is the same sort of equilibrium point for (N) as $(0,0)$ is for (L).

Example 8.6.

$$\left.\begin{aligned}\dot{x} &= -x^3\\ \dot{y} &= -y^3\end{aligned}\right\} \qquad (N)$$

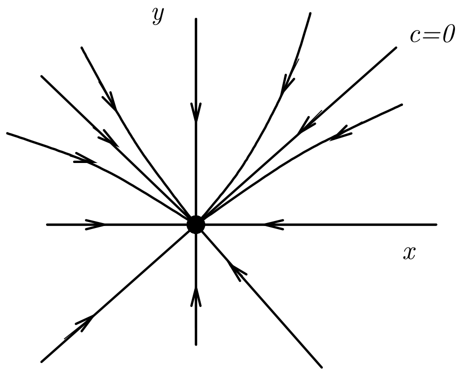
The linearised equation of (N) at $(0,0)$ is

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus the eigenvalue is $\lambda = 0$, and we can say nothing about this equilibrium point. However, we can compute the trajectories directly from

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y^3}{x^3}.$$

Integrating, we find that the trajectories are $y^2 = x^2/(1 + cx^2)$ and (N) has the phase plane



$(0,0)$ is a stable equilibrium point.

Example 8.7.

$$\left.\begin{aligned} \dot{x} &= y - x\sqrt{x^2+y^2} \\ \dot{y} &= -x - y\sqrt{x^2+y^2} \end{aligned}\right\} \qquad (N)$$

The linearised equation at $(0,0)$ is

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}\right\} \qquad (L)$$

i.e.

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{x}(t) = A\boldsymbol{x}(t) \qquad (L)$$

λ is an eigenvalue of $A \Leftrightarrow \lambda^2 + 1 = 0$ i.e. $\lambda = \pm i$.
Hence $(0,0)$ is a centre for the linearised equation and so we can draw no conclusion about (N).
Suppose we think about distance squared (x^2+y^2) from $(0,0)$ of a solution of (N). We have

$$\begin{aligned} \dot{x}x &= xy - x^2\sqrt{x^2+y^2} \\ \dot{y}y &= -xy - y^2\sqrt{x^2+y^2} \end{aligned}$$

and hence $\frac{d}{dt}(x^2+y^2) = 2\dot{x}x + 2\dot{y}y = -2(x^2+y^2)^{\frac{3}{2}} < 0$

$$\Rightarrow x^2 + y^2 = \frac{1}{(t+c)^2}$$

$$\left\{ \frac{dV}{dt} = -2V^{\frac{3}{2}} \Rightarrow -\frac{1}{2}\frac{dV}{dt} = V^{\frac{3}{2}} \Rightarrow \int -\frac{1}{2}V^{-\frac{3}{2}}dV = \int dt \Rightarrow V = \frac{1}{(t+c)^2} \right\}$$

Thus if $\boldsymbol{x}$ is any solution of (N), x^2+y^2 is a decreasing function of t, and so if $\boldsymbol{x}(0)$ is close to $(0,0)$, then $\boldsymbol{x}(t)$ gets closer to $(0,0)$ for all t, i.e. $(0,0)$ is a stable equilibrium point of (N).

Example 8.8.

$$\left.\begin{aligned} \dot{x} &= y + x\sqrt{x^2+y^2} \\ \dot{y} &= -x + y\sqrt{x^2+y^2} \end{aligned}\right\} \qquad (N)$$

Repeating the above analysis we obtain

$$\frac{d}{dt}(x^2 + y^2) = 2(x^2 + y^2)^{\frac{3}{2}} > 0$$

and so $(0, 0)$ is an unstable equilibrium point.

The above technique is often useful in establishing whether an equilibrium point is stable or not. The function $x^2 + y^2$ can be replaced by any function V with similar properties.

Theorem 8.9. *Consider the system*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t)) \qquad (N)$$

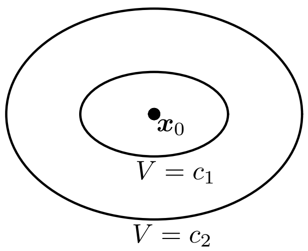
with equilibrium point $\boldsymbol{x}_0$. Let U be an open set containing $\boldsymbol{x}_0$ and let $V : U \to \mathbb{R}$ be a smooth function such that

(i) $\boldsymbol{x}_0$ is a minimum point for V

(ii) $\frac{d}{dt}(V(\boldsymbol{x}(t))) \leq 0$ whenever $\boldsymbol{x}$ is a solution of N

Then $\boldsymbol{x}_0$ is a stable equilibrium point.

Proof. Since $\boldsymbol{x}_0$ is a local minimum for V, the level curves of V, i.e., curves with equations of the form $V(x, y) = c$, form a system of closed curves surrounding $\boldsymbol{x}_0$. When $c = V(\boldsymbol{x}_0)$ the level curves reduces to the point $\boldsymbol{x}_0$, and as c increases the level curve moves further from $\boldsymbol{x}_0$.



$c_2 > c_1 > V(\boldsymbol{x}_0)$

Since $\dfrac{d}{dt}(V(\boldsymbol{x}(t))) \leq 0$, if $\boldsymbol{x}(0)$ lies inside $V(\boldsymbol{x}) = c$, i.e., if $V(\boldsymbol{x}(0)) < c$, then $V(\boldsymbol{x}(t)) < c$ for all $t > 0$. Hence, the trajectory will stay inside $V(\boldsymbol{x}) = c$ for all $t > 0$. It follows that $\boldsymbol{x}_0$ is a stable equilibrium point.

□

Remarks

(a) If condition (ii) in the statement of the theorem is changed to $\frac{d}{dt}(V(\boldsymbol{x}(t)) < 0$ for every solution $\boldsymbol{x}(t)$, then the conclusion becomes *$\boldsymbol{x}_0$ is an asymptotically stable equilibrium point.*

(b) If condition (ii) in the statement of the theorem is changed to $\frac{d}{dt}(V(\boldsymbol{x}(t)) > 0$ for every solution $\boldsymbol{x}(t)$, then the conclusion becomes $\boldsymbol{x}_0$ *is an unstable equilibrium point.*

The functions V above are called Lyapunov functions - they provide the means of determining the stability of an equilibrium point when the Linearisation theorem (connection between (N) and (L)) cannot be applied.

In general it is difficult to find a Lyapunov function for a given equation - however it is sometimes possible.

Example 8.10.

(i) To see if $V(x,y) = x^2 + y^2$ is a Lyapunov function for

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

Compute $\frac{dV}{dt} = 2xf(x,y) + 2yg(x,y)$. We need this to be always positive, always negative, or zero.
See examples above.

(ii) If the equations of the trajectories can be found explicitly these may give Lyapunov functions.
e.g. Consider the pendulum equation $\ddot{x} + \sin(x) = 0$.
The equivalent system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) \end{aligned}$$

The trajectories are given by

$$\frac{1}{2}y^2 - \cos x = constant.$$

(This equation has the interpretation of 'total energy = kinetic energy + potential energy = constant'.)
Let $V(x,y) = \frac{1}{2}y^2 - \cos x$. Clearly, V has a minimum at $(x,y) = (0,0)$.

$$\begin{aligned} \text{Also } \frac{d}{dt}(V(x(t),y(t))) &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \\ &= \sin x \times y + y \times (-\sin x) = 0 \end{aligned}$$

Hence $(0,0)$ is a stable equilibrium point.

(iii) Consider the system

$$\left.\begin{array}{c}\dot{x}(t)=-x-2y^2\\ \dot{y}(t)=xy-y^3\end{array}\right\}\qquad (N)$$

The corresponding linearised equation at $(0,0)$ is

$$\left.\begin{array}{c}\dot{x}=-x\\ \dot{y}=0\end{array}\right\}\qquad (L)$$

i.e.

$$\dot{\boldsymbol{x}}(t)=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\boldsymbol{x}(t)=A\boldsymbol{x}(t)\qquad (L)$$

and so the Linearisation theorem linking (N) and (L) does not apply.

We seek a Lyapunov function of the form $V(x,y)=ax^2+by^2$ where a and b are positive constants.
Then V has a minimum at $(0,0)$

$$\begin{aligned}\frac{d}{dt}V(x(t),y(t)) &= \frac{\partial V}{\partial x}\frac{dx}{dt}+\frac{\partial V}{\partial y}\frac{dy}{dt}\\ &= 2ax(-x-2y^2)+2by(xy-y^3)\\ &= -2ax^2+(2b-4a)xy^2-2by^4\end{aligned}$$

If we choose $a=1, b=2$ we obtain

$$\frac{d}{dt}V(x(t),y(t))=-2x^2-4y^4<0\qquad (\boldsymbol{x}(t)\neq(0,0))$$

Hence $V(x,y)=x^2+2y^2$ is a Lyapunov function and $(0,0)$ is an asymptotically stable equilibrium point.