B Revision of Matrices

We consider real valued $n \times n$ matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ & \vdots & \\ a_{n1} & a_{n2} & \cdots a_{nn}. \end{pmatrix}, \qquad a_{ij} \in \mathbb{R} \tag{B.4}$$

Facts

- A^{-1} exists iff $\det(A) \neq 0$, in which case we say that A is **non-singular** (if $\det(A) = 0$, then A is said to be **singular**).
- A non-zero vector $\boldsymbol{v}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ ($\lambda \in \mathbb{C}/\{0\}$) is called an **eigenvector**, and λ is called an **eigenvalue**.
- λ is an eigenvalue iff $\det(A - \lambda\mathbb{I}) = 0$.
- $\det(A - \lambda\mathbb{I}) = 0$ is an n-th order polynomial equation with n roots $\lambda_1, \cdots, \lambda_n$, which are either real or come in complex conjugate pairs (from the fundamental theorem of algebra).
- The eigenvectors corresponding to a complex conjugate pair of eigenvalues (λ, λ^*) are also a conjugate pair $(\boldsymbol{v}, \boldsymbol{v}^*)$ (since taking the complex conjugate of $A\boldsymbol{v} = \lambda\boldsymbol{v}$ gives $A\boldsymbol{v}^* = \lambda^*\boldsymbol{v}^*$).
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- If a given eigenvalue is repeated m times, then it is said to have **algebraic multiplicity** m. Each distinct eigenvalue has at least one associated eigenvector, and an eigenvalue with algebraic multiplicity m may have q linearly independent eigenvectors with $1 \leq q \leq m$.
- It follows from the above two points that a $n \times n$ matrix with n distinct eigenvalues has n linearly independent eigenvectors.
- Real symmetric matrices (i.e., with $a_{ij} = a_{ji}$) have:
  1. real eigenvalues
  2. n linearly independent eigenvectors $\boldsymbol{y}^{(1)}, \cdots \boldsymbol{y}^{(n)}$- regardless of the algebraic multiplicities
  3. if all eigenvalues are distinct (i.e. all have algebraic multiplicity one), then the eigenvectors form an orthogonal set, i.e. $\boldsymbol{y}^{(i)}.\boldsymbol{y}^{(j)} = 0$ for $i \neq j$.