Question 1 (14 Marks)
Find the general solutions of the following first order differential equations.

(a) $\dfrac{dy}{dx} - 2xy = 6xe^{x^2}$,

(b) $(\sin x + x^2 e^y - 1)\dfrac{dy}{dx} + y\cos x + 2xe^y = 0$.

Question 2 (8 Marks)
Consider the following first order differential equation

$$\frac{dy}{dx} = y - y^3. \qquad (*)$$

(a) Find all constant solutions and hence sketch the direction field of $(*)$.

(b) Sketch the solution of $(*)$ satisfying the initial condition $y(0) = \frac{1}{2}$ in the direction field you made for (a) (there is no need to solve the initial value problem explicitly).

Question 3 (12 Marks)
Consider the Euler equation

$$x^2\frac{d^2y}{dx^2} + (2\alpha + 1)x\frac{dy}{dx} + \alpha^2 y = 0,$$

where α is a real constant.

(a) Find the general solution of the equation.

(b) Find the condition on α so that the general solution approaches zero as $x \to \infty$.

Question 4 (16 Marks)
Solve the following initial value problems

(a) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = e^{2t} \quad y(0) = 0,\ y'(0) = 1,$

(b) $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 \quad y(0) = 1,\ y'(0) = 1.$

continued overleaf

Question 5 (12 Marks)
An object of mass 1 kg is attached to a spring with spring constant 4 Nm^{-1}. The spring is immersed in a viscous liquid with damping constant 1 N sec m^{-1} and an external force $F(t) = 2\cos\omega t$ N is applied to the mass.

(a) Write down the differential equation governing the motion of the object.

(b) Explain (without first solving the equation) what is meant by the transient solution and the steady state solution of the system.

(c) Find the steady state solution and give its amplitude A as a function of ω.

(d) Define the term resonance and find the value of the amplitude A at the resonance frequency.

Question 6 (12 Marks)
Consider the second order differential equation

$$x^2y'' - 3xy' + (4x+4)y = 0.$$

(a) Show that $x = 0$ is a regular singular point of the equation.

(b) Use the method of Frobenius to find a nonzero solution of the equation.

Question 7 (16 Marks)
Using the method of Laplace transforms solve the following initial value problems

(a) $y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1,$ where $u_2(t)$ is given by

$$u_2(t) = \begin{cases} 0 & \text{if} \quad t < 2 \\ 1 & \text{if} \quad t \geq 2, \end{cases}$$

(b) $y'(t) = t + \int_0^t y(t-\tau)\cos(\tau)\,d\tau, \qquad y(0) = 4.$

Question 8 (10 Marks)
(a) Construct the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} = f(x) \qquad y(0) + y'(0) = 0,\ y(1) - y'(1) = 0.$$

(b) Define the Dirac delta-function $\delta(s-x)$ and show that the Green's function found in (a) satisfies

$$\frac{\partial^2 G}{\partial s^2}(x,s) = \delta(s-x).$$

End of paper