1. Find the general solution of each of the following differential equations

(*i*) $(xe^y - 2y)\,\dfrac{dy}{dx} + e^y + 2x = 0.$

(*ii*) $\dfrac{dy}{dx} - y = y^3.$ **[14 marks]**

2. Show that there can be no solution to the initial value problem

$$(y - x)\,\frac{dy}{dx} + y^2 = 0; \quad y(1) = 1$$

on any open interval containing $x = 1$ and explain why this does not contradict Picard's Theorem. **[5 marks]**

3. An industrial company increases in value at a rate proportional to the square root of its current value. If the company was worth £1 million two years ago and is worth £2 million today, determine when it will be worth £3 million.

Determine also how long ago the company was worth only £100,000. **[7 marks]**

4. Solve the initial value problem

$$\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 1; \quad y(0) = 1; \quad \frac{dy}{dx}(0) = 0.$$

[7 marks]

5. Verify that $y(x) = e^{-x}$ is a solution of

$$x\,\frac{d^2y}{dx^2} + (2x - 1)\,\frac{dy}{dx} + (x - 1)\,y = 0$$

and hence find a fundamental set of solutions for the equation.

State Abel's Theorem and verify that it is satisfied by this fundamental set of solutions. **[15 marks]**

continued overleaf

6. Find the general solution of the equation

$$x\,\frac{d^2y}{dx^2} + (x-1)\,\frac{dy}{dx} - y = 0$$

by first finding a suitable factorization for the differential expression.

[10 marks]

7. An object of mass 1 kg is attached to a spring with spring constant 2 N/m and is immersed in a viscous fluid which is such that the damping constant is 3 Nm/sec. Write down the differential equation which describes the motion of the mass and find its general solution.

The mass is at rest in its equilibrium position at time $t = 0$. It is then subjected to a downwards impulse of 1 Ns. Determine the subsequent motion of the mass. **[8 marks]**

8. By using the method of Frobenius find a non-zero series solution of the differential equation

$$x^2\,\frac{d^2y}{dx^2} + 4x\,\frac{dy}{dx} + (x+2)\,y = 0.$$

[11 marks]

9. (*i*) By using the method of Laplace transforms solve

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + y = t^2e^{-t}; \qquad y(0) = 3, \quad \frac{dy}{dt}(0) = -1.$$

(*ii*) If $g(x) = \int_0^x f(t)\,dt$, show that the Laplace transforms of f and g satisfy

$$\overline{g}(s) = \frac{\overline{f}(s)}{s}.$$

[9 marks]

10. Find a fundamental set of solutions for the differential equation

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2\,y = 0.$$

Hence find a Green's function for the boundary value problem

$$x^2\,\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 2\,y = f(x); \qquad y(1) \;=\; 0 \;=\; y(2).$$

[14 marks]

END OF PAPER