1. Find the general solution of the differential equation

$$\frac{dy}{dx} - y = e^{-2x}$$

Draw the direction field for the equation and sketch the graphs of some typical solutions of the equation. **[10 marks]**

2. Show that there can be no solution to the initial value problem

$$y\frac{dy}{dx} + x^2 + y^2 = 0; \quad y(1) = \alpha$$

on any open interval containing $x = 1$ when $\alpha = 0$

For what values of α does the initial value problem have a solution in an interval containing $x = 1$? Give a brief justification for your answer.

[7 marks]

3. Solve the following initial value problem

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = \mathrm{e}^{2x}, \qquad y(0) = -1, y'(0) = 2.$$

[9 marks]

4. By using the method of variation of parameters find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 9y = \frac{1}{\sin 3x}.$$

[11 marks]

5. Find the fundamental set of solutions of the following differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0.$$

State the Abel's Theorem and verify that it applies in the situation above.

[8 marks]

6. It is found that the mass m immersed in a viscous fluid and hanging in

equilibrium stretches the spring by the distance l. The damping constant r corresponds exactly to the case of critical damping. The mass is subsequently pulled down an additional distance x_0 and released with the upward velocity v_0. Find the maximum initial velocity $v_0^{\max}$ such that the mass does not cross the equilibrium position $x = 0$.

[11 marks]

7. Use the method of Frobenius to determine a nonzero solution of the following differential equation

$$4x^2 \frac{d^2y}{dx^2} + (3x+1)y = 0.$$

[12 marks]

8. Using the method of Laplace transform solve the following initial value problems

(i)

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2 \mathrm{e}^{3t}, \quad y(0) = 2, y'(0) = 6,$$

(ii)

$$\frac{d^2y}{dt^2} + y = \delta(t - 2\pi), \quad y(0) = y'(0) = 0.$$

Sketch the graph of solution in the case (ii).

[15 marks]

9. Find the function $y(t)$ if

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)d\tau.$$

[6 marks]

10. Find the Green's function of the following boundary value problem

$$\frac{d^2y}{dx^2} = f(x), \quad y(0) + 3y'(0) = 0, \quad y(1) + y'(1) = 0.$$

Use this result to solve $\dfrac{d^2y}{dx^2} = x^2$ subject to the same boundary conditions.

[11 marks]

END OF PAPER