1. Find the solution of the following differential equation

$$(1-x)\frac{dx}{dt} = \mathrm{e}^{t-x}$$

satisfying the initial condition $x(0) = 0$.

[9 marks]

2. Find all constant solutions of the equation

$$\frac{dy}{dx} = y^2 - 1$$

Sketch the graphs of the solutions of the equation satisfying the initial condition (i) $y(0) = 0$ and (ii) $y(0) = 2$.

[7 marks]

3. Show that the following equation is exact and find its general solution

$$\mathrm{e}^x + y + (x - 2\sin y)\frac{dy}{dx} = 0.$$

[10 marks]

4. Find the general solution of the following equation

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 6y = 0.$$

State Abel's Theorem and verify that it holds in the situation above.

[11 marks]

5. Using the variation of parameters method find the general solution of the following equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = \sin(\mathrm{e}^t).$$

[12 marks]

6. An object of mass m is attached to a spring with spring constant k.

The spring-mass system is then immersed in a viscous liquid with damping constant r and an external force $F(t) = A \sin \omega t$ is applied to the mass. Explain what is meant by (i) the transient solution and (ii) the steady-state (asymptotic) solution for the system. Find the steady-state solution and show that it has the amplitude $R = \frac{A}{\sqrt{r^2\omega^2 + (k - m\omega^2)^2}}$.

[11 marks]

7. Use the method of Frobenius to find a nonzero solution of the following equation

$$x^2 \frac{d^2y}{dx^2} + (x^2 + \frac{1}{4})y = 0.$$

[12 marks]

8. Use the Laplace transform method to solve the following initial value problem

$$\frac{d^2y}{dt^2} - y = 2\delta(t-1), \qquad y(0) = 1, y'(0) = 0.$$

[9 marks]

9. Solve for $x(t)$ using the convolution theorem

$$x(t) + \int_0^t x(t-\tau)\mathrm{e}^{-\tau} d\tau = 1.$$

[9 marks]

10. Find a Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} + y = f(x), \quad y(0) + y'(0) = 0, \; y(\pi) = 0, \quad 0 \le x \le \pi,$$

and hence write down a solution for the problem.

[10 marks]

END OF PAPER