Question 1 (22 Marks)
Find the general solutions of the following differential equations

(a) $\dfrac{dy}{dt} + (\tan t)y = 0,$

(b) $x\cos(xy)\dfrac{dy}{dx} + y\cos(xy) + 2x = 0,$

(c) $\dfrac{d^2y}{dx^2} = 2y\left(\dfrac{dy}{dx}\right)^3.$

Question 2 (12 Marks)
Consider the following first order differential equation

$$\frac{dy}{dx} = x - y. \qquad (*)$$

(a) Determine the regions in the xy-plane where $(i)\ \dfrac{dy}{dx} < 0,\quad (ii)\ \dfrac{dy}{dx} = 0,\quad (iii)\ \dfrac{dy}{dx} > 0.$

(b) Sketch the direction field of $(*)$.

(c) Using an appropriate integrating factor, find the solution of $(*)$ satisfying the initial condition $y(0) = -1$ and sketch it in the direction field you made for (b).

Question 3 (8 Marks)
Consider the second order linear homogeneous differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

where a_1 and a_0 are continuous functions. Prove that the space of solutions of this equation is a vector space of dimension 2. You may assume without proof that appropriate initial value problems have unique solutions.

Question 4 (12 Marks)
Using the method of variation of parameters find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{(e^x)}.$$

continued overleaf

Question 5 (14 Marks)
An object of mass 1 kg is attached to a spring with spring constant 10 N/m and is immersed in a viscous fluid with damping constant 2 Nm/s.

(a) Write down the differential equation which describes the motion of the mass and find its general solution.

(b) The mass is at rest and in its equilibrium position at time $t = 0$. At time $t = 1$ s it is subjected to a downward impulse of 1 Ns. Using the method of Laplace transforms determine the subsequent motion of the mass.

Question 6 (12 Marks)
Consider the differential equation

$$xy'' + (1 - x)y' - y = 0.$$

(a) Show that $x = 0$ is a regular singular point of the equation.

(b) Use the method of Frobenius to find a non-zero solution. Is the solution analytic at $x = 0$?

Question 7 (8 Marks)

Use the Convolution Theorem to solve the following equation for $y(t)$

$$3 \sin 2t = y(t) + \int_0^t y(t - \tau)\, \tau \, d\tau.$$

Question 8 (12 Marks)
(a) Find the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} + y = f(x), \qquad y(0) = 0, \quad y'(\pi) = 0.$$

(b) Use the result of (a) to find the solution of

$$\frac{d^2y}{dx^2} + y = x, \qquad y(0) = 0, \quad y'(\pi) = 0.$$

End of paper