Question 1 (14 Marks)
Find the general solutions of the following differential equations for $x > 0$

(a) $y' + \dfrac{y}{x} = 3\cos x,$

(b) $2y \ln x \dfrac{dy}{dx} + \dfrac{y^2}{x} + 2x = 0.$

Question 2 (13 Marks)
Solve the following initial value problems for $y(x)$

(a) $y\dfrac{dy}{dx} + x = 0,\ y(0) = 1,$

(b) $y'' + y = e^x,\ y(0) = 1,\ y'(0) = 0.$

Question 3 (14 Marks)
Consider the second order linear inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x),$$

where a_1, a_0 and f are continuous functions. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the homogeneous equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Using the method of variation of the parameters show that a particular solution of the inhomogeneous equation is given by

$$y(x) = \int_0^x \frac{(y_1(t)y_2(x) - y_2(t)y_1(x))\, f(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}\, dt.$$

Question 4 (14 Marks)
An object of mass $m > 0$ is attached to a spring with spring constant $k > 0$ and is immersed in a viscous fluid with damping constant $r > 0$.

(a) Write down the differential equation which describes the motion of the mass.

(b) In terms of m, k and r, explain what is meant by underdamping, overdamping and critical damping.

(c) Consider the case $m = 1$ kg, $r = 2$ N s m^{-1} and $k = 1$ N m^{-1}. At time $t = 0$ the object is raised 1 m and given an initial velocity of 2 m s^{-1} in the downward direction. Solve the equation of motion with these initial conditions and sketch the position of the object as a function of time.

continued overleaf

Question 5 (14 Marks)
Use the method of Frobenius to find a non-zero solution of the following differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$$

Question 6 (18 Marks)
Using the method of Laplace transforms solve the following equations for $y(t)$

(a) $y'' + y' - 6y = 2e^{3t}, \qquad y(0) = 0, \quad y'(0) = 0,$

(b) $e^{-t} = y(t) + 2\int_0^t y(t-\tau)\cos(\tau)\,d\tau.$

Question 7 (13 Marks)
(a) Define the Wronskian of two functions y_1 and y_2. Use Abel's theorem to show that the Wronskian of any two solutions of the homogeneous linear equation

$$\frac{d^2y}{dx^2} - y = 0.$$

is constant.

(b) Find the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} - y = f(x), \qquad y(0) = 0, \quad y'(1) = 0.$$

End of paper