F13YT2 June 07 SOLN

Q1 a) (i) This is homogenous. Letting $u = \frac{y}{x}$

gives $\quad x u' + \cancel{u} = \cancel{u} - 2\frac{1}{u} \quad$ s.t.

$$u'u = -\frac{2}{x} \quad \text{and} \quad \frac{u^2}{2} = -2\ln(x) + c$$

Thus $\quad y^2 = x^2\left(-4\ln(x) + 2c\right)$ [redacted]

(ii) This is exact since

$$\frac{\partial}{\partial x}\left(x^2\cos(y)\right) = 2x\cos(y) = \frac{\partial}{\partial y}\left(2x\sin(y) + 6e^{2x}\right)$$

Soln given by $\psi(x,y) = c$ where.

$$\psi(x,y) = \int x^2\cos(y)\,dy = x^2\sin(y) + g(x)$$

& $\psi(x,y) = \int \left(2x\sin(y) + 6e^{2x}\right)dx$

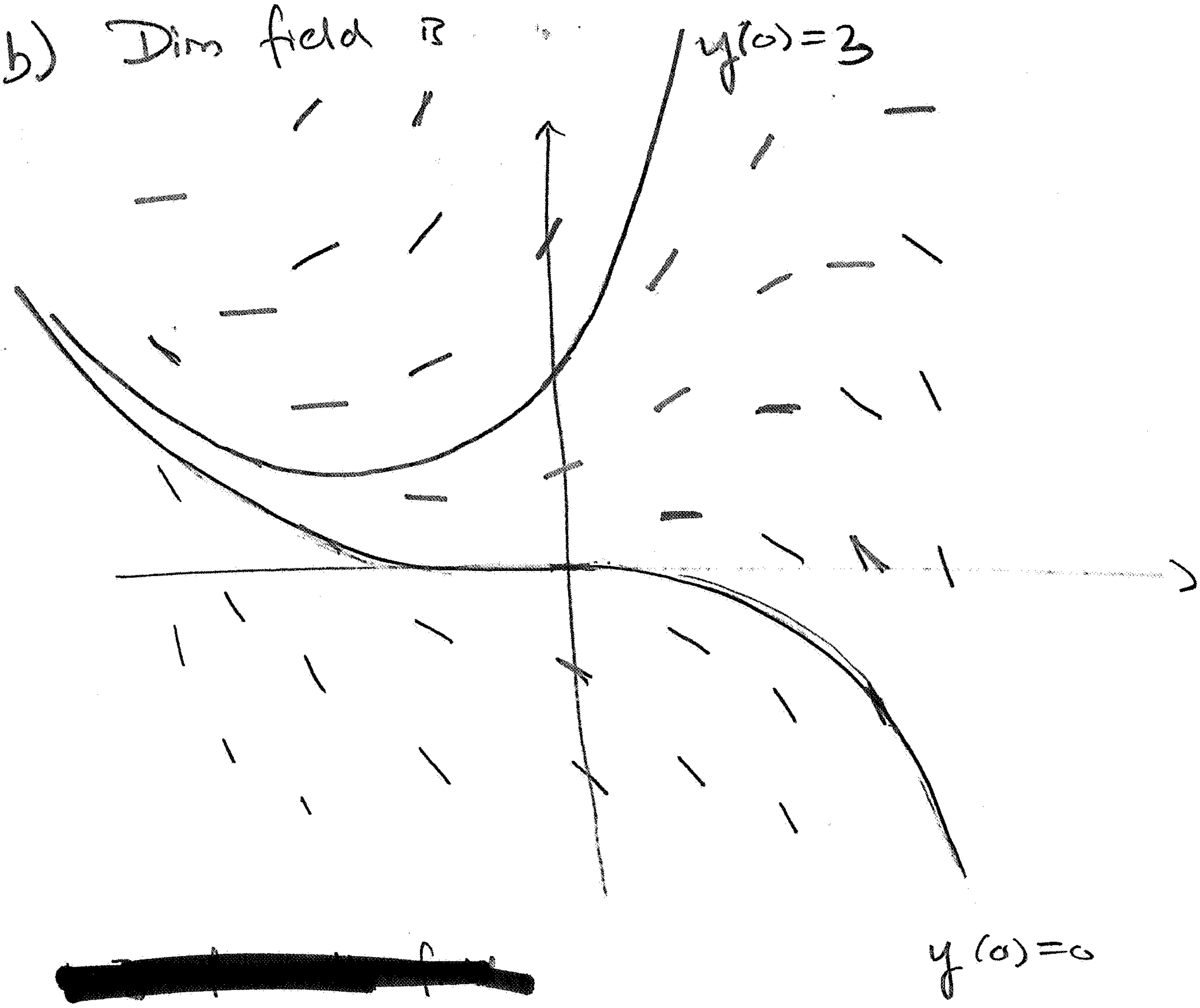
$$= x^2\sin(y) + 3e^{2x} + h(y)$$

Thus $\psi(x,y) = x^2\sin(y) + 3e^{2x}$ and

soln given implicitly by

$$x^2\sin(y) + 3e^{2x} = c$$ [redacted]

b) Dim field B



[redacted]

[redacted]

Q2 (a) Identifying $x = x_1$.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ -5x_1 - 2x_2^2 - 3x_3 \end{pmatrix}$$ [redacted]

(b) Write as $\dot{\underline{x}} = A\underline{x}$, where

$A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$ has eigenvalues $(-1 \pm i)$

and eigenvectors $\begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$ [2] [or $\begin{pmatrix} 5 \\ 2 \mp i \end{pmatrix}$]

Thus a complex soln is $\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$

$= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t} (\cos(t) + i\sin(t))$ [redacted]

$= e^{-t} \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + ie^{-t} \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}$

Thus we have gen soln:

$$\underline{x} = c_1 e^{-t} \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$$

$$+ c_2 e^{-t} \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}$$ [redacted]

[or $c_1 e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos(t) + \sin(t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$]

Q3 The characteristic eqn is $(\lambda^2+\lambda-2)=0$

which factorises as $(\lambda-1)(\lambda+2)=0$.

Thus we have 2 solns $y^{(1)}(t)=e^{t}$, $y^{(2)}(t)=e^{-2t}$

of homogeneous eqn. [redacted]

The Wronskian is

$$W(y^{(1)};y^{(2)}) = e^{-t}(-2-1) = -3e^{-t}$$ [redacted]

and the variation of parameters soln is

$$x_p(t) = C_1(t)\,y^{(1)}(t) + C_2(t)\,y^{(2)}(t)$$ with

$$C_1(t) = -\int \frac{y^{(2)}(t)\cdot 3e^{2t}\,dt}{W(y^{(1)}(t), y^{(1)}(t))} = -\int \frac{3\,dt}{-3e^{-t}} = e^{t}$$

$$C_2(t) = +\int \frac{y^{(1)}(t)\,3e^{2t}\,dt}{W(y^{(1)}(t), y^{(2)}(t))}\,dt = \int \frac{3e^{3t}\,dt}{-3e^{-t}} = -\frac{e^{4t}}{4}$$

Thus $x_p(t) = e^{2t} - \frac{1}{4}e^{+2t} = \frac{3}{4}e^{2t}$ [redacted]

([redacted])

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4. a) Using reduction of order, let $z = y'$, such that $z' + 3z = e^{2x}$. This is linear [redacted] with int factor e^{3x}. Thus

$$ze^{3x} = \int e^{5x}\,dx = \frac{e^{5x}}{5} + c$$

and so $z = \frac{e^{2x}}{5} + c$.

$$z(0) = y'(0) = \frac{1}{5} = \frac{1}{5} + c \Rightarrow c = 0.$$

Thus $\frac{dy}{dx} = \frac{e^{2x}}{5}$ and $y = \frac{e^{2x}}{10} + d$.

$$y(0) = \frac{11}{10} = \frac{1}{10} + d \Rightarrow d = 1$$ and

so $y(x) = \frac{e^{2x}}{10} + 1$ [redacted]

b) This is of Euler type. [redacted] With subst $u = \ln(x)$ we find

$$\frac{d^2y}{du^2} + 3\frac{dy}{du} - 10u = 0$$

which has characteristic eqn

$$(\lambda^2 + 3\lambda - 10) = (\lambda+5)(\lambda-2) = 0$$

Thus

$$y(x) = Ae^{-5u} + Be^{2u}$$
$$= Ax^{-5} + Bx^{2}$$ [redacted]

Q5. (i) let $\bar{y}(s) = \dfrac{s}{(s^2-2s-3)} = \dfrac{s}{(s-3)(s+1)}$.

Then letting $\bar{y}(s) = \dfrac{A}{(s-3)} + \dfrac{B}{(s+1)}$

$\Rightarrow \quad s = A(s+1) + B(s-3)$

letting $s=-1 \Rightarrow -1 = -4B \Rightarrow B = 1/4$

letting $s=3 \Rightarrow 3 = 4A \Rightarrow A = 3/4$

and $\bar{y}(s) = \dfrac{3/4}{(s-3)} + \dfrac{1/4}{(s+1)}$. [redacted]

Inverting $y(t) = 3/4\, e^{3t} + \frac{1}{4} e^{-t}$ [redacted]

(ii) let $\bar{y}(s) = \dfrac{e^{-3s} s}{(s^2+2s+5)}$

$$= \frac{e^{-3s} s}{(s+1)^2+4} = \frac{e^{-3s}(s+1)}{(s+1)^2+4} - \frac{1}{2} e^{-3s} \frac{2}{(s+1)^2+4}$$

$$= e^{-3s} \mathcal{L}[e^{-t}\cos(2t)] - \frac{1}{2} e^{-3s} \mathcal{L}[e^{-t}\sin(2t)]$$

[redacted]

Thus

$$y(t) = u_3(t)\, e^{-(t-3)} \cos(2(t-3))$$
$$- \frac{1}{2} u_3(t)\, e^{-(t-3)} \sin(2(t-3)).$$ [redacted]

b) Taking LT gives

$$\left(s^2\bar{y}(s) - s y(0) - \dot{y}(0)\right) + 4\left(s\bar{y}(s) - y(0)\right) + 4\bar{y}(s) = \frac{4}{s}$$

Subs $y(0) = 1$, $\dot{y}(0) = 0$ gives.

$$\bar{y}(s)\left(s^2+4s+4\right) - (4+s) = \frac{4}{s}$$ [redacted]

or $$\bar{y}(s) = \frac{4+s}{(s+2)^2} + \frac{4}{s(s+2)^2}$$

writing $$\frac{4}{s(s+2)^2} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$$

we have $$4 = A(s+2)^2 + B\,s(s+2) + C s$$

letting $s=0 \Rightarrow 4 = 4A \Rightarrow A = 1$.

letting $s=-2 \Rightarrow 4 = -2C \Rightarrow C = -2$.

Coeff of $s^2 \Rightarrow A + B = 0 \Rightarrow B = -1$

[2] for Par. Fra.

Thus

$$\bar{y}(s) = \frac{1}{(s+2)} + \frac{2}{(s+2)^2} + \frac{1}{s} - \frac{1}{(s+2)} - \frac{2}{(s+2)^2}$$

$$= \frac{1}{s}$$

Thus $y(t) = 1$. [redacted]

Q6 Two linearly indep solns of homogeneous eqn are $u_1(x) = \sin(2x)$, $u_2(x) = \cos(2x)$ [redacted]

$y^{(1)}(x) = \sin(2x)$ satisfies $y^{(1)}(0) = 0$

$y^{(2)}(x) = \cos(2x)$ satisfies $y^{(2)}(\pi) = 0$. [redacted]

The Wronskian is $W(y^{(1)}(x), y^{(2)}(x)) = -2\sin^2(2x) - 2\cos^2(2x)$

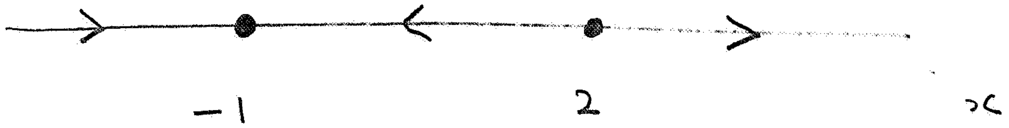
$= -2$. [redacted]

Thus the Green's fn is:

$$G(x,s) = \begin{cases} -\frac{1}{2}\sin(2s)\cos(2x) & 0 \leq s \leq x \leq \pi \\ -\frac{1}{2}\sin(2x)\cos(2s) & 0 \leq x \leq s \leq \pi \end{cases}$$

[redacted]

Q7 (a) Writing as $\dot{x} = (x-2)(x+1)$ we obtain phase plane:



[redacted]

(b) Write as $\dot{x} = y$, $\dot{y} = x^3$.

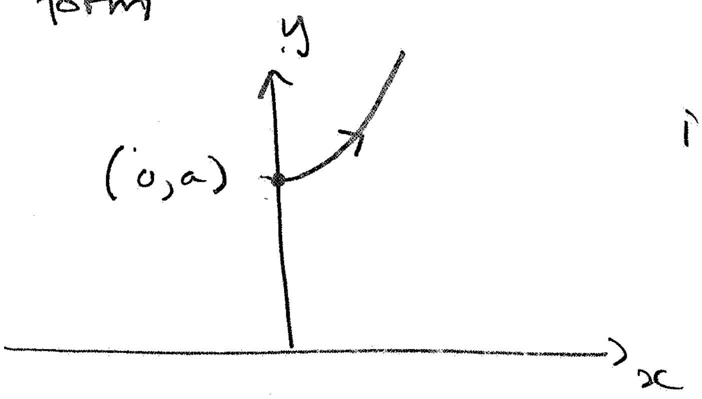
The only equib pt is $(0,0)$

We integrate $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x^3}{y}$.

Integrating gives $\frac{y^2}{2} = \frac{x^4}{4} + c$, or

$y^2 = \frac{1}{2}x^4 + c$. [redacted]

let us consider the traj through $(0,a)$, $a \geq 0$ given by $y^2 = \frac{1}{2}x^4 + a^2$. This will be of form

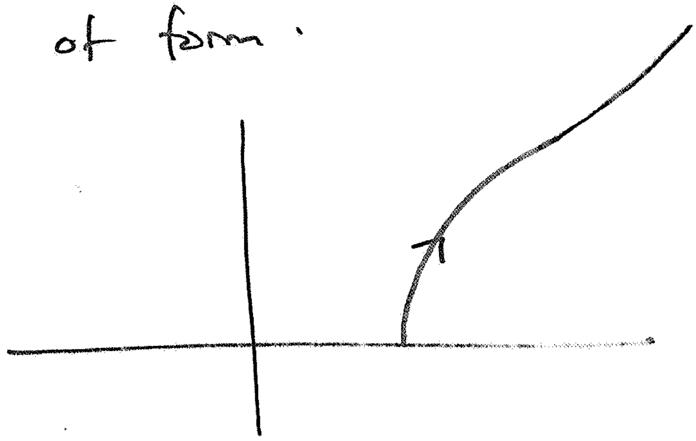


in 1st quadrant.

Traj through $(b,0)$, $b>0$, will be

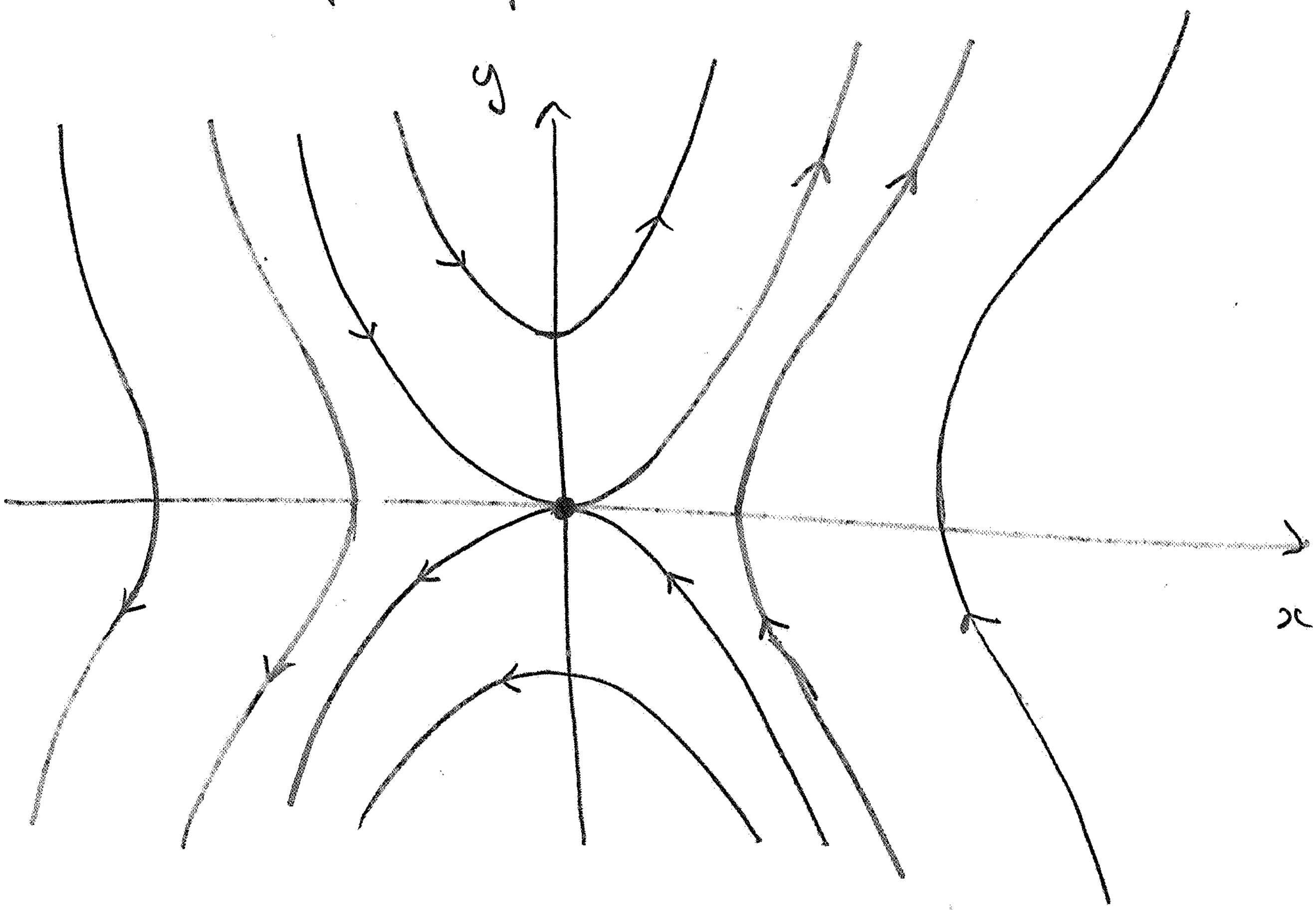
$y^2 = \frac{1}{2}x^4 + \frac{1}{2}b^4$, which will

be of form:



Using $x \leftrightarrow -x$, $y \to -y$ symmetry

we have phase plane.

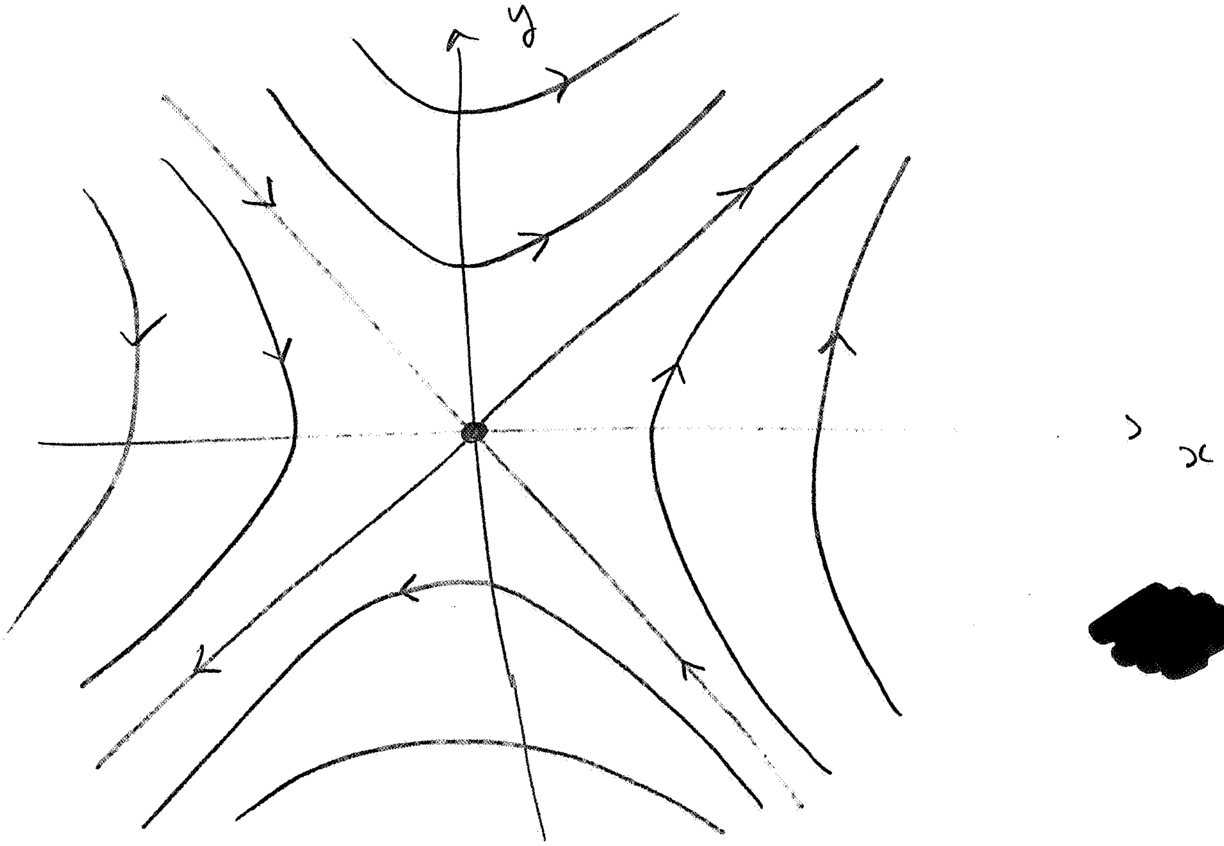


[redacted]

Q8 $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ has eigenvalues $-3, 1$

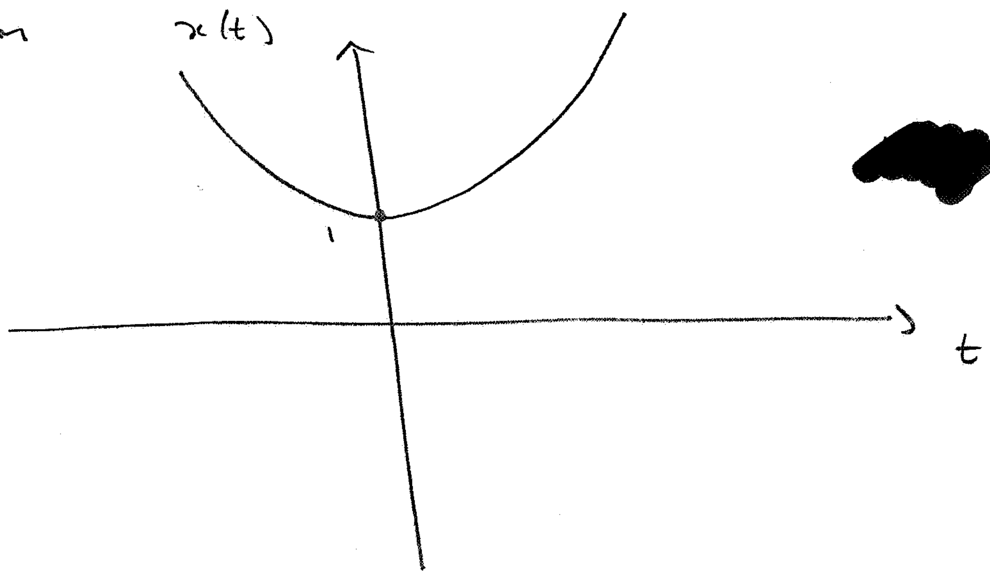
and corresponding eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. [redacted]

Thus the equib. point $(0,0)$ is a saddle point [2], and we have phase plane.



[redacted]

The soln is of form



[redacted]

9. $$\frac{dV}{dt} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= 2ax(2xy^2 + 4x^2y^2) + 2by(-x^3y + y^3)$$

$$= 4ax^2y^2 + 2by^4 + x^3y^2(8a - 2b)$$ [redacted]

Thus if we choose $a, b > 0$ and $8a - 2b = 0$, then $V(x,y)$ has a global minimum at $(0,0)$ & $\frac{dV}{dt} \geq 0$; and so $(0,0)$ is an unstable point.

$a = 1$, $b = 4$ is one such choice, resulting in

$$V(x,y) = x^2 + 4y^2.$$ [redacted]

[redacted]