Formulae sheets are at the end of the examination paper

1. a) Find the general solution of the following differential equations

$$(i)\ \frac{dy}{dx} = (1-x^2)y^2 \qquad (ii)\ \frac{dy}{dx} + \frac{y}{x} = e^x \qquad (iii)\ x\cos(xy)\frac{dy}{dx} + y\cos(xy) + 2x = 0$$

b) Consider the ordinary differential equation

$$\frac{dy}{dx} = y^2 - 4 \tag{1}$$

(i) Sketch the direction field of (1).

(ii) Sketch the solution satisfying $y(0) = 1$. **[12 marks]**

2. a) Find the general solution of the following 2nd order ODEs

$$(i)\ y'' - 2y' + y = x + x^2, \quad (ii)\ x^2y'' + 2xy' - 2y = 0.$$

b) Use the method of reduction of order to find the solution of the initial value problem

$$y'y'' - 2x^3 = 0, \quad y'(1) = 1, \ \ y(1) = 1.$$

[12 marks]

3. Use the method of variation of parameters to find a particular solution of the equation

$$\frac{d^2y}{dx^2} + y = 1 + \sec(x)$$

[You may use the integral $\int \tan(x)\,dx = -\ln(\cos(x))$] **[10 marks]**

4. a) Find the inverse Laplace transforms of the following functions

$$(i)\ \frac{-3}{s^2+s-2} \quad (ii)\ \frac{1}{s^3} + \frac{(s+3)e^{-s}}{s^2+2s+5}$$

b) Use Laplace transforms to solve the following initial value problem

$$\ddot{y} + 2\dot{y} + 2y = \delta(t-2), \ \ y(0) = 1, \ \ \dot{y}(0) = 0$$

c) Define the convolution of two functions $f, g : [0, \infty) \to \mathbb{R}$. State and prove the convolution theorem for Laplace transforms. **[15 marks]**

continued ...

5. Solve the following boundary value problem by using an appropriate Green's function:

$$\frac{d^2y}{dx^2} = x^2, \quad y(0) = 0, \quad y(2) = 0.$$

[10 marks]

6. Consider the boundary value problem

$$-y'' = \lambda y, \quad y(0) = 0, \quad y(1) + 2y'(1) = 0.$$

Show that this problem has infinitely many eigenvalues (considering each of the possibilities $\lambda < 0$, $\lambda = 0$, $\lambda > 0$) and find the corresponding eigenfunctions. **[10 marks]**

7. Write the equation

$$\ddot{x} = x - x^3$$

as a system of 1st order equations and find all the equilibrium points and the equation of the trajectories in the phase plane.

Hence sketch the phase plane for the system. **[11 marks]**

8. Consider the linear systems

$$(i) \quad \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$(ii) \quad \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

In each case, determine the nature of the equilbrium point in the phase plane and sketch the phase plane.

For (i), sketch $x(t)$ for the solution satisfying the conditions $x(0) = 1$, $y(0) = 0$.

[12 marks]

9. By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, determine whether $(0, 0)$ is a stable or an unstable equilibrium point for the system

$$\dot{x} = x^5 + 3xy, \quad \dot{y} = -x^2 + y^3.$$

[8 marks]

END OF PAPER

Table of Standard Laplace Transforms

$y(t)$	$\bar{y}(s)$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
$e^{\alpha t}$	$\dfrac{1}{s-\alpha}$
$\cos(\alpha t)$	$\dfrac{s}{s^2+\alpha^2}$
$\sin(\alpha t)$	$\dfrac{\alpha}{s^2+\alpha^2}$
$f'(t)$	$s\bar{f}(s) - f(0)$
$f''(t)$	$s^2\bar{f}(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$e^{\alpha t}f(t)$	$\bar{f}(s-\alpha)$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}\bar{f}(s)$
$\delta(t-c)$	e^{-cs}
$f * g(t)$	$\bar{f}(s)\,\bar{g}(s)$