F13YT2 June 06 SOLNS

1. (i) This is separable, and

$$\int \frac{dy}{y^2} = \int (1-x^2)\,dx \quad \text{s.t.} \quad -\frac{1}{y} = x - \frac{x^3}{3} + c$$

Hence $$\boxed{y = \frac{-1}{\left(x - \frac{x^3}{3} + c\right)}} \qquad \left(= \frac{3}{(x^3 - 3x + c)}\right)$$

(ii) This is linear, with int factor

$$e^{\ln(x)} = x, \quad \text{s.t.}$$

$$\frac{d}{dx}(yx) = xe^x, \quad \text{and} \quad yx = \int xe^x\,dx$$

So $$yx = xe^x - \int e^x\,dx = e^x(x-1) + c$$

Hence $$\boxed{y = e^x(1-x^{-1}) + cx^{-1}}$$

(iii) This is exact since:

$$\frac{\partial(x\cos(xy))}{\partial x} = \cos(xy) - xy\sin(xy)$$

$$= \frac{\partial(y\cos(xy) + 2x)}{\partial y}$$

Soln given by $\varphi(x,y) = c$, where:

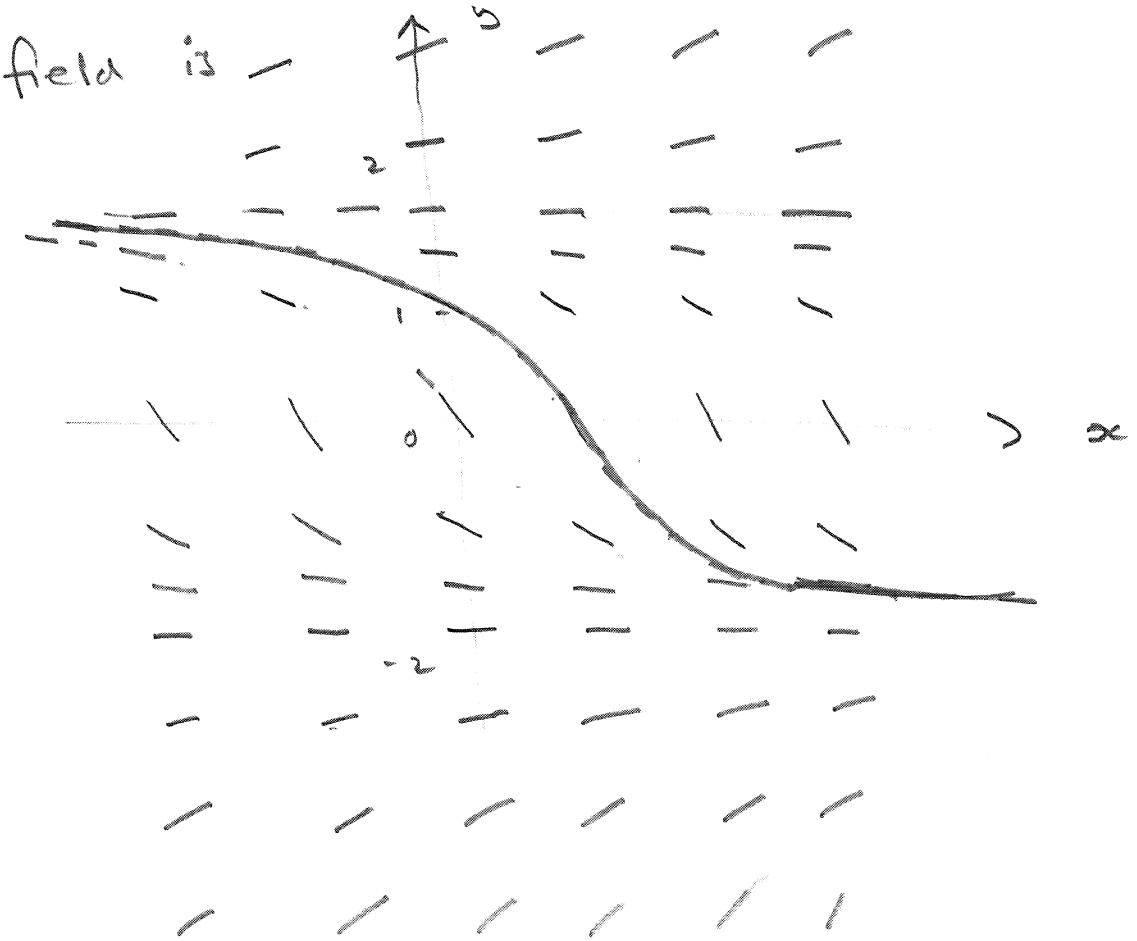
$$\varphi(x,y) = \int x\cos(xy)\,dy = \sin(xy) + f(x)$$

$$= \int (y\cos(xy) + 2x)\,dx = \sin(xy) + x^2 + g(y)$$

Hence $\varphi(x,y) = \sin(xy) + x^2$ and gen soln given implicitly by $\boxed{\sin(xy) + x^2 = c.}$

(b) We have $\frac{dy}{dx} = (y-2)(y+2)$.

Dir field is



Soln satisfying $y(0) =$ shown

2. a) (i) First find FSS

Char eqn is $(\lambda^2 - 2\lambda + 1) = (\lambda - 1)^2 = 0$

Hence, complementary fn is

$y(x) = Ae^x + Bxe^x$.

Guess particular soln of form

$y_p(x) = \alpha + \beta x + \gamma x^2$

Substituting into eqn we find

$y_p(x) = 8 + 5x + x^2$ and gen soln is $\boxed{y(x) = Ae^x + Bxe^x + 8 + 5x + x^2}$

(ii) This is of Euler type. letting

$u = \ln(x)$, we have $x\frac{d}{dx} = \frac{d}{du}$

$x^2\frac{d^2}{dx^2} = \frac{d^2}{du^2} - \frac{d}{du}$ and so eqn becomes

$$\frac{d^2y}{du^2} + \frac{dy}{du} - 2y = 0$$

Char eqn $(\lambda^2 + \lambda - 2) = (\lambda + 2)(\lambda - 1) = 0$

Hence $\boxed{y = Ax^{-2} + Bx}$.

2. b) let $z = y'$, i.e.

$z z' = 2x^3$.

Integrating gives $\frac{z^2}{2} = \frac{x^4}{2} + c$

or $z^2 = x^4 + c$

$z(1) = 1 \Rightarrow c = 0$

Hence $z^2 = x^4$, or $z = x^2$ (since $z(1) > 0$)

Hence $y' = x^2$, and integrating

gives $y = \frac{x^3}{3} + d$

$y(1) = 1 \Rightarrow d = \frac{2}{3}$

Hence $y(x) = \frac{1}{3}(x^3 + 2)$.

3. First find FSS $y^{(1)}(x) = \cos(x)$

$y^{(2)}(x) = \sin(x)$ of homo eqn

$$y'' + y = 0.$$

Then var. of param. tells us that

$$y_p(x) = c_1(x) y^{(1)}(x) + c_2(x) y^{(2)}(x).$$

where $$c_1(x) = -\int \frac{y^{(2)}(x)\,(1+\sec(x))}{W(y^{(1)}(x), y^{(2)}(x))}\,dx$$

$$c_2(x) = \int \frac{y^{(1)}(x)\,(1+\sec(x))}{W(y^{(1)}(x), y^{(2)}(x))}\,dx.$$

In our case $W(y^{(1)}(x), y^{(2)}(x)) = 1$

& $$c_1(x) = -\int (\sin(x) + \tan(x))\,dx$$

$$= \cos(x) + \ln(\cos(x))$$

$$c_2(x) = \int (\cos(x) + 1)\,dx = \sin(x) + x$$

Thus $$y_p(x) = \cos^2(x) + \cos(x)\ln(\cos(x)) + \sin^2(x) + x\sin(x)$$

$$= 1 + x\sin(x) + \cos(x)\ln(\cos(x))$$

4 a) (i) Write as

$$\frac{-3}{(s-1)(s+2)} = \frac{-1}{(s-1)} + \frac{1}{(s+2)}$$

Hence inv trans $= -e^{t} + e^{-2t}$

(ii) Rewrite 2nd term as:

$$\frac{(s+1)e^{-s}}{(s+1)^2+4} + \frac{2e^{-s}}{(s+1)^2+4}$$

Hence, whole expression

$$= \frac{1}{2}\mathcal{L}[t^2] + e^{-s}\mathcal{L}[e^{-t}\cos(2t)] + e^{-s}\mathcal{L}[e^{-t}\sin(2t)]$$

$\mathcal{L}$ inv trans $= \frac{1}{2}t^2 + u_1(t)e^{-(t-1)}\cos(2(t-1))$

$$+ u_1(t)e^{-(t-1)}\sin(2(t-1))$$

b) Taking Lapl trans gives:

$$(s^2+2s+2)\bar{y}(s) - sy(0) - \dot{y}(0) - 2y(0) = e^{-2s}$$

So $\bar{y}(s)\left((s+1)^2+1\right) = (s+2) + e^{-2s}$

$$\bar{y}(s) = \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1} + \frac{e^{-2s}}{(s+1)^2+1}$$

$$\overline{y}(s) = \mathcal{L}[\sin(t)e^{-t}] + \mathcal{L}[\cos(t)e^{-t}] + e^{-2s}\mathcal{L}[\sin(t)e^{-t}]$$

Hence $$y(t) = \sin(t)e^{-t} + \cos(t)e^{-t} + u_2(t)\sin(t-2)e^{-(t-2)}$$

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c) The convolution is defined by

$$f*g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau.$$

The Conv Thm states $\overline{f*g}(s) = \overline{f}(s)\cdot\overline{g}(s)$

Proof $$\overline{f*g}(s) = \int_0^\infty e^{-st} f*g(t)\,dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right)dt$$

$$= \int_0^\infty g(\tau)\left(\int_\tau^\infty f(t-\tau)\,e^{-st}\,dt\right)d\tau \qquad (\dagger)$$

Making the subst $v = t-\tau$

$$(\dagger) = \int_0^\infty g(\tau)\left(\int_0^\infty f(v)e^{-sv}\,dv\right)e^{-s\tau}\,d\tau$$

$$= \overline{f}(s)\cdot\overline{g}(s) \qquad \square$$

5. First find the FSS of homo eqn.

$u_1(x) = 1$, $u_2(x) = x$. Hence, we can choose $y^{(1)}(x) = x$, $y^{(2)}(x) = x-2$ to satisfy $y^{(1)}(0) = 0$, $y^{(2)}(2) = 0$.

The Wronskian is

$$W(y^{(1)}(x), y^{(2)}(x)) = x \cdot 1 - 1 \cdot (x-2) = 2.$$

Hence, the Green's fn is:

$$G(x,s) = \begin{cases} \frac{s(x-2)}{2} & 0 \le s \le x \le 2 \\ \frac{x(s-2)}{2} & 0 \le x \le s \le 2 \end{cases}$$

The soln is $y(x) = \int_0^2 G(x,s)\, s^2\, ds$.

$$= \int_0^x s^3\, ds \ \frac{(x-2)}{2} + \frac{x}{2} \int_x^0 (s-2)\, s^2\, ds$$

$$= \left(\frac{x-2}{2}\right)\frac{x^4}{4} + \frac{x}{2}\left[\frac{s^4}{4} - \frac{2s^3}{3}\right]_x^2$$

$$= \frac{x^5}{8} - \frac{x^4}{4} + \frac{x}{2}\left(\frac{2^4}{4} - \frac{2^4}{3}\right) - \frac{x}{2}\left(\frac{x^4}{4} - \frac{2x^3}{3}\right)$$

$$= x^4\left(\frac{1}{3} - \frac{1}{4}\right) + x\,2^3\left(\frac{1}{4} - \frac{1}{3}\right)$$

$$= \frac{1}{12}\left(x^4 - 8x\right) = \frac{x^4}{12} - \frac{2}{3}x$$

6. let us consider the different cases

$\lambda < 0$ letting $\lambda = -k^2$, we have

$y'' = k^2 y$ with gen soln.

$y = A \sinh(kx) + B \cosh(kx)$.

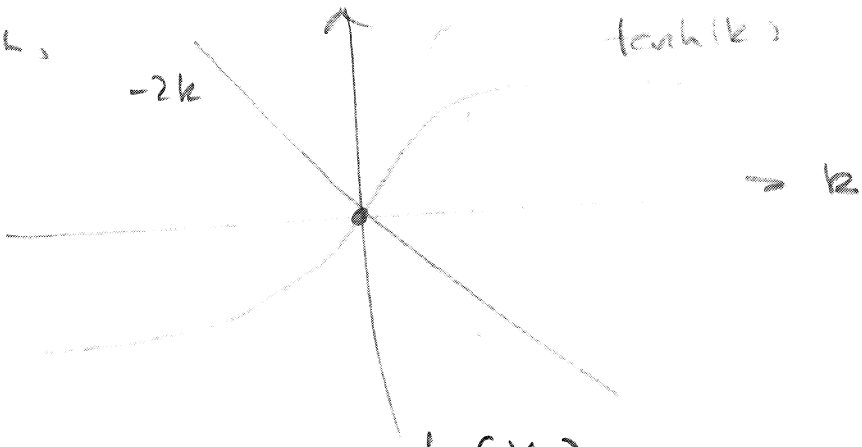
$y(0) = 0 \iff B = 0$.

$y(1) + 2y'(1) = 0 \iff A \sinh(k) + 2kA \cosh(k) = 0$

So either $A = 0$ or

$\tanh(k) = -2k$. (*)

From the graph,



there are no non-zero solns of (*).

Hence no eigenvals $\lambda < 0$

$\lambda = 0$ Gen soln $y = A + Bx$

$y(0) = 0 \iff A = 0$

$y(1) + 2y'(1) = 0 \iff B + 2B = 0$

Only soln is $y = 0$. and hence no eigenvals.

$\underline{\lambda > 0}$ letting $\lambda = k^2$, we have:

$y'' = -k^2 y$ with gen soln:

$y = A\sin(kx) + B\cos(kx)$.

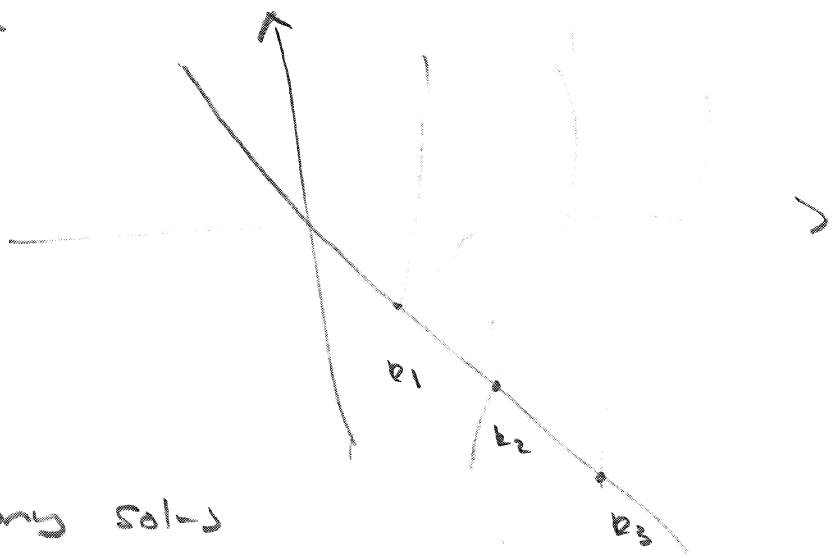
$y(0) = 0 \iff B = 0$

$y(1) + 2y'(1) = 0 \iff A\sin(k) + 2Ak\cos(k) = 0$

so either $A = 0$ or

$\tan(k) = -2k$. (+)

From the graph



there are as many solns $k_1, k_2, \ldots$ to (+)

The corres eigenfns are

$y(x) = \sin(k_n x)$

7. The system is

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$

Equib points are $(x,y) = (0,0)$ and $(1,0)$, $(-1,0)$.

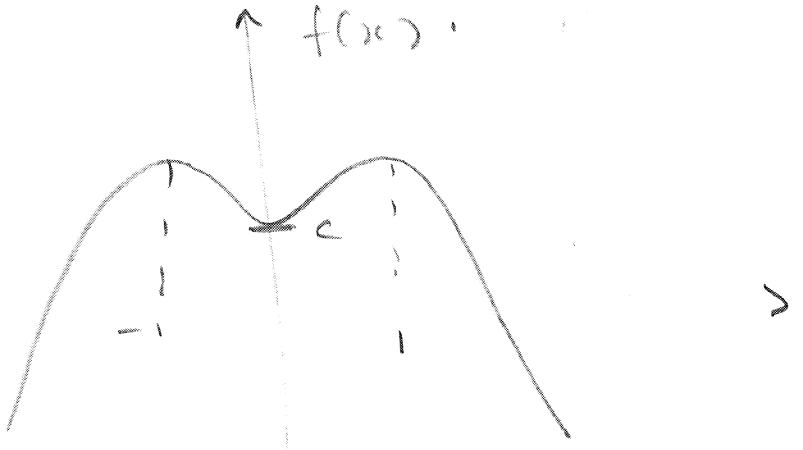
The trajectory $y(x)$ obeys

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x - x^3}{y}$$

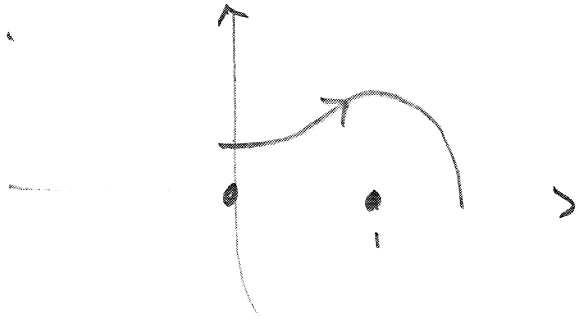
Integrating gives $\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^4}{4} + c$

or $y^2 = x^2 - \frac{x^4}{2} + c$

We have $y^2 = f(x)$, where $f(x)$ is of form



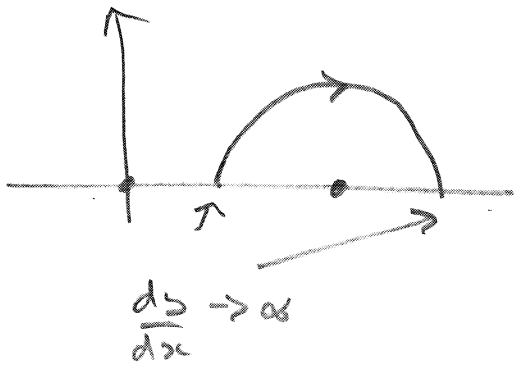
It is enough to consider traj in 1st quadrant of phase plane. A traj through $(0,a)$ always increase with x, reaches a max at $x = +1$ & then decreases ie.



7 cont

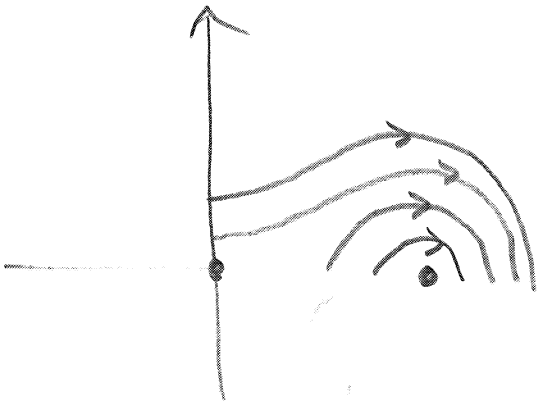
Similarly. A traj through $(b, 0)$, $0 < b < 1$, increases, reaches a max at $x = 1$ & decreases.

We have:

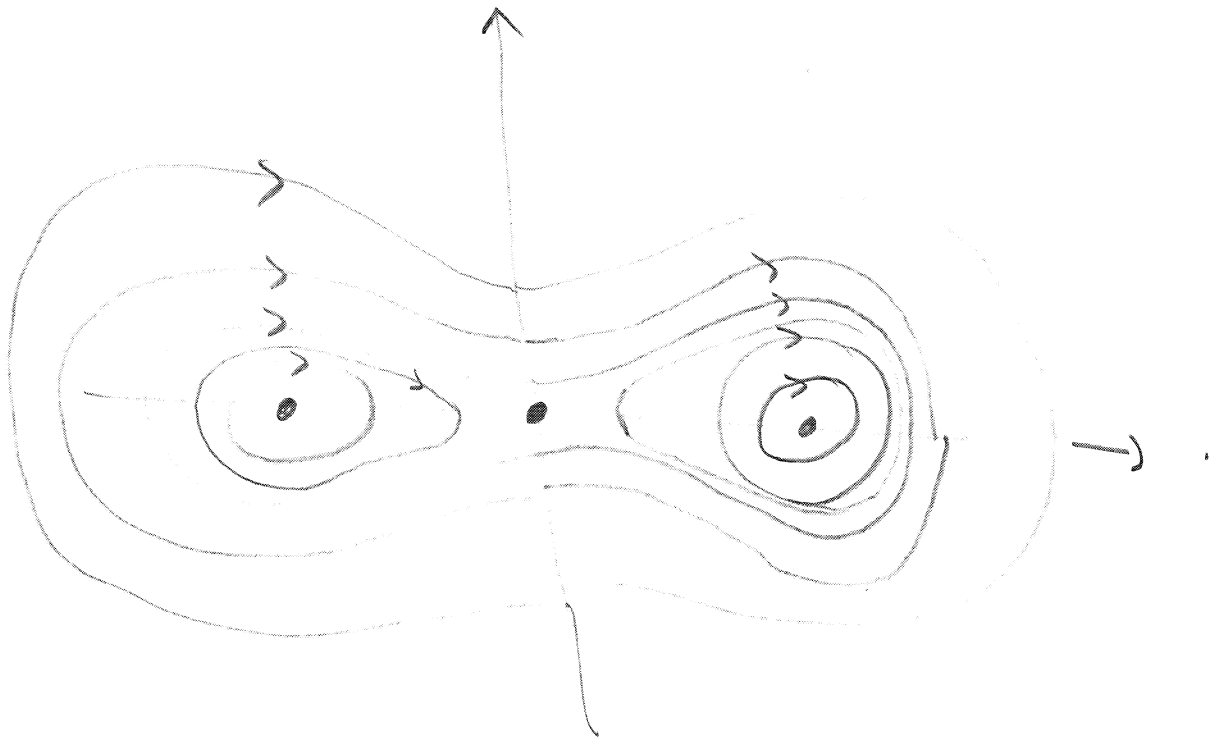


$\frac{dy}{dx} \to \infty$

Overall, have



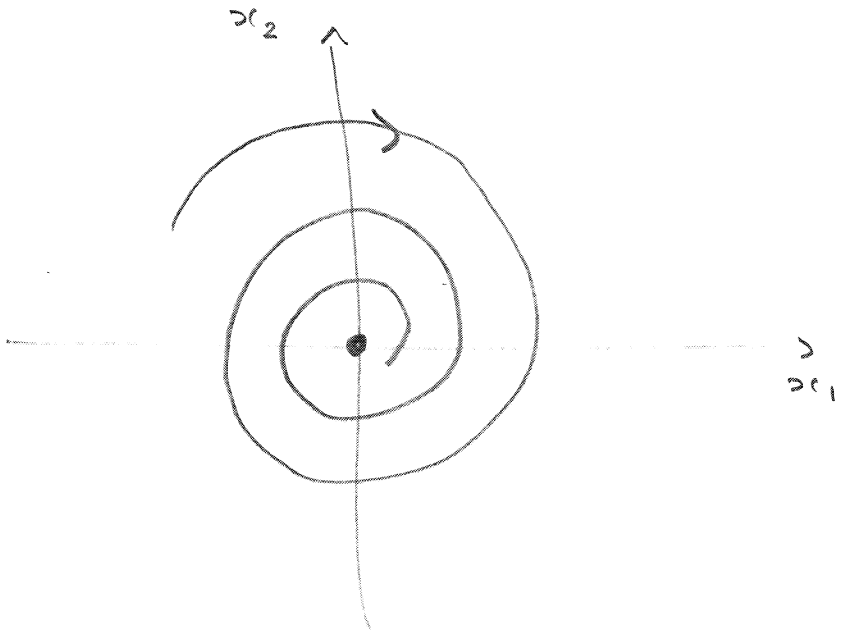
& by symmetry, overall phase plane is:



8. (i) Eigenvalues $-1 \pm 2i$. Hence

use $(0,0)$ is a <u>stable spiral point</u> and

phase plane is

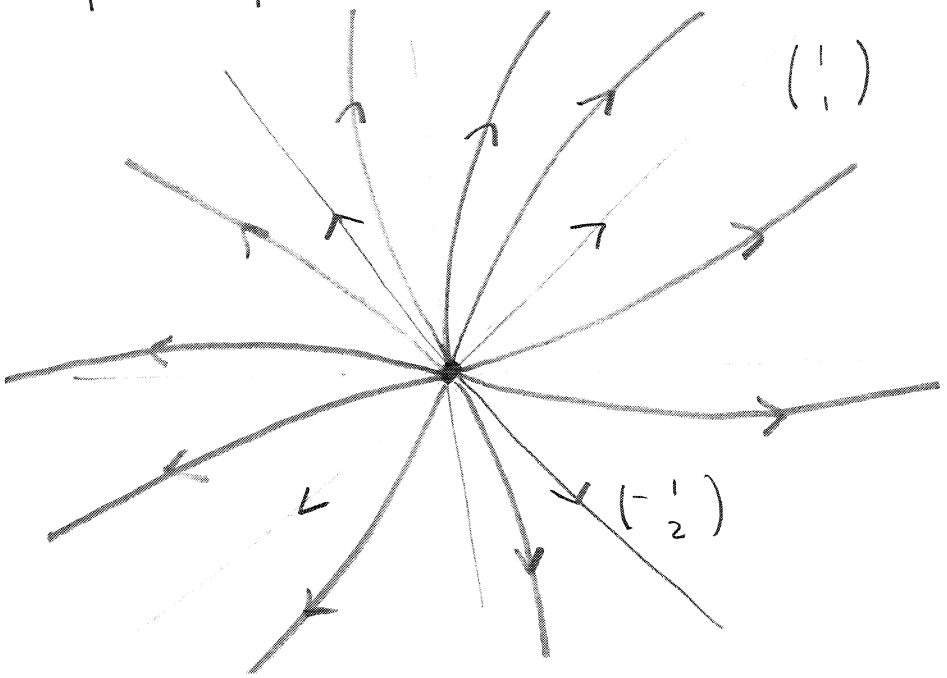


Clockwise since $\dot{x}_1 = x_2$ - hence arrows to right in upper half plane

(ii) Eigenvals $2, 5$, Eigenvector $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

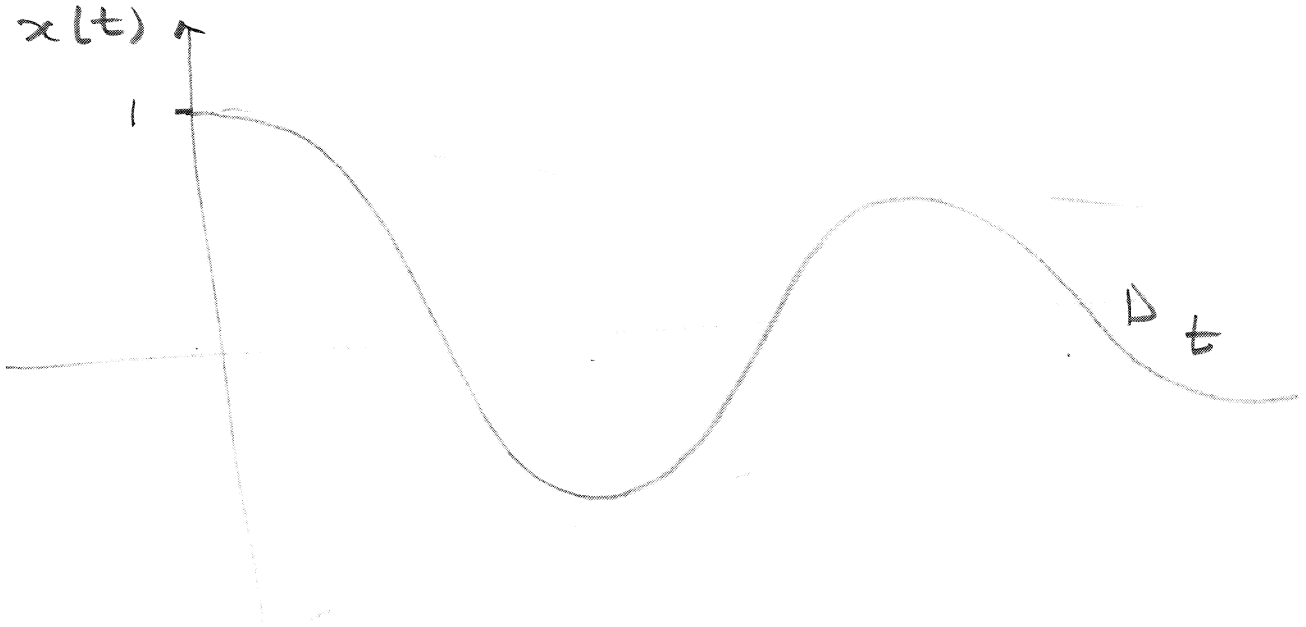
<u>Unstable node</u>.

Hence, phase plane is



8 cont

For (i) $x(t)$ with $x(0) = 1$ is



9. let $V = ax^2 + by^2$

Then $\frac{dV}{dt} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$

$= 2ax(x^5 + 3xy)$

$+ 2by(-x^2 + y^3)$

$= 2ax^6 + 2by^4 + x^2y(6a - 2b)$

Choose $a = 1$, $b = 3$ given.

$V = x^2 + 3y^2$, which has a min at (0,0) and

$\frac{dV}{dt} = 2x^6 + 6y^4 \geq 0$. Hence

(0,0) is an unstable equib point.