Formulae sheets are at the end of the examination paper

1. a) Find the general solution of the following differential equations

$$(i)\ \frac{dy}{dx} = \frac{y}{x} - 2\,\frac{x}{y} \qquad (ii)\ x^2\cos(y)\frac{dy}{dx} + 2x\sin(y) + 6e^{2x} = 0$$

b) Consider the ordinary differential equation

$$\frac{dy}{dx} = y - x^2 \tag{1}$$

(i) Sketch the direction field of (1).

(ii) Sketch the solutions satisfying $y(0) = 3$ and $y(0) = 0$. **[12 marks]**

2. a) Write the following 3rd order ODE as a system of first order ODEs

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 2\left(\frac{dx}{dt}\right)^2 + 5x = 0.$$

b) Find the general solution of the following system of equations by computing the eigenvalues and eigenvectors of an appropriate matrix

$$\frac{dx_1}{dt} = x_1 - 5x_2, \qquad \frac{dx_2}{dt} = x_1 - 3x_2.$$

[11 marks]

3. Use the method of variation of parameters to find a particular solution of the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 3e^{2t}.$$

[10 marks]

4. a) Find the solution of the following initial value problems

$$y'' + 3y' = e^{2x}, \quad y'(0) = \frac{1}{5}, \ y(0) = \frac{11}{10}.$$

b) Find the general solution of the equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 10y = 0.$$

[12 marks]

5. a) Find the inverse Laplace transforms of the following functions

$$(i)\ \frac{s}{s^2 - 2s - 3}, \qquad (ii)\ \frac{e^{-3s}s}{s^2 + 2s + 5}.$$

b) Use Laplace transforms to solve the following initial value problem

$$\ddot{y} + 4\dot{y} + 4y = 4, \ y(0) = 1, \ \dot{y}(0) = 0$$

[15 marks]

Paper continues ...

6. Find the Green's function associated with the following boundary value problem:

$$\frac{d^2y}{dx^2} + 4y = f(x), \quad y(0) = 0, \quad y'(\pi) = 0.$$

[10 marks]

7. a) Draw the phase plane of the first order equation

$$\frac{dx}{dt} = x^2 - x - 2.$$

b) Write the equation

$$\frac{d^2x}{dt^2} = x^3$$

as a system of 1st order equations. Then, find the equilibrium point or points and the equation of the trajectories in the phase plane. Hence, sketch the phase plane.

[12 marks]

8. Consider the linear system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Determine the nature of the equilbrium point in the phase plane and sketch the phase plane.

Sketch $x(t)$ for the solution satisfying the conditions $x(0) = 1$, $y(0) = 0$.

[10 marks]

9. By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, determine whether $(0, 0)$ is a stable or an unstable equilibrium point for the system

$$\dot{x} = 2xy^2 + 4x^2y^2, \quad \dot{y} = -x^3y + y^3.$$

[8 marks]

END OF PAPER

Table of Standard Laplace Transforms

$y(t)$	$\bar{y}(s)$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
$e^{\alpha t}$	$\dfrac{1}{s-\alpha}$
$\cos(\alpha t)$	$\dfrac{s}{s^2+\alpha^2}$
$\sin(\alpha t)$	$\dfrac{\alpha}{s^2+\alpha^2}$
$f'(t)$	$s\bar{f}(s) - f(0)$
$f''(t)$	$s^2\bar{f}(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$e^{\alpha t}f(t)$	$\bar{f}(s-\alpha)$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}\bar{f}(s)$
$\delta(t-c)$	e^{-cs}
$f * g(t)$	$\bar{f}(s)\,\bar{g}(s)$