Sample Exam - 3 hours - Attempt all Questions

1. (a) Find the general solution of the following differential equations

$$(i)\ x\frac{dy}{dx} = e^{-y} \quad (ii)\ \frac{dy}{dx} + 2y = \cosh(x) \quad (iii)\ \frac{dy}{dx} - 4y = y^4.$$

(b) Plot the direction field of the equation

$$\frac{dy}{dx} = \frac{1}{4}x^2 + y^2 - 4.$$

Hence, sketch the solution satisfying $y(0) = 0$. **[12 marks]**

2. (a) Find the solution of the following initial value problems

$$(i)\ y'' = 6y^5, \quad y(0) = 1, \quad y'(0) = \sqrt{2}$$
$$(ii)\ x^2y'' + 4xy' + 2y = 0, \quad y(1) = 2, \quad y'(1) = -3$$

(b) Find the general solution of

$$y'' - 3y' + 2y = \cosh(x)$$

using the method of variation of parameters. **[13 marks]**

3. Find the general solution of the systems of equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad (1)$$

Hence find the solution of the initial value problem consisting of (1) and the initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

[11 marks]

4. Use Laplace transforms to solve the following initial value problem

$$\ddot{y} + 2\dot{y} + 5y = 3e^{-t}\sin(t), \quad y(0) = 0, \quad \dot{y}(0) = 3$$

[11 marks]

continued ...

5. Solve the following initial value problem by finding an appropriate Green's function

$$\frac{d^2y}{dx^2} + y = x, \quad y(0) = 0, \;\; y'(\pi) = 0.$$

[10 marks]

6. Discuss the existence of eigenvalues of the boundary value problem

$$-y'' = \lambda y, \quad 2y(0) + y'(0) = 0, \;\; y(1) = 0$$

in each of the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. **[10 marks]**

7. Write the equation

$$\ddot{x} = x - x^2$$

as a system of 1st order equations and find all the equilibrium points and the equations of the trajectories in the phase plane.

Hence sketch the phase plane for the system.
Also sketch the graph of the solution $x(t)$ of the equation satisfying $x(0) = 1/2$, $\dot{x}(0) = 0$. **[13 marks]**

8. Find all equilibrium points of the system

$$\dot{x} = x - xy - x^2; \quad \dot{y} = -y - 2xy$$

and determine the nature of each. Sketch a possible phase plane. **[12 marks]**

9. By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, determine whether $(0, 0)$ is a stable or unstable equilibrium point for the system

$$\dot{x} = x^3 - y^3; \qquad \dot{y} = 2xy^2 + y.$$

[8 marks]

END OF PAPER