VT2 SAMPLE PAPER

1. (a) (i) Separating, we have

$$\int e^y\,dy = \int x\,dx \quad , \text{ with soln } \quad e^y = \left(\frac{x^2}{2} + c\right)$$

Thus $$y = \ln\left(\frac{x^2}{2} + c\right)$$

(ii) This is linear with int factor e^{2x}. Thus

$$e^{2x} y = \int e^{2x} \cosh(x)\,dx = \frac{1}{2}\int (e^{3x} + e^{x})\,dx$$

$$= \frac{1}{2}\left(\frac{e^{3x}}{3} + e^{x}\right) + c = \frac{1}{6}\left(e^{3x} + 3e^{x}\right) + c$$

Hence gen soln is $$y(x) = \frac{1}{6}\left(e^{x} + 3e^{-x}\right) + c$$

(iii) This is off Bernoulli type. Multiplying by y^{-4} gives

$$y^{-4}\frac{dy}{dx} - 4y^{-3} = 1. \quad \text{Subst } u = y^{-3}, \text{ we have}$$

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = -3y^{-4}\frac{dy}{dx}. \quad \text{Thus the eqn becomes}$$

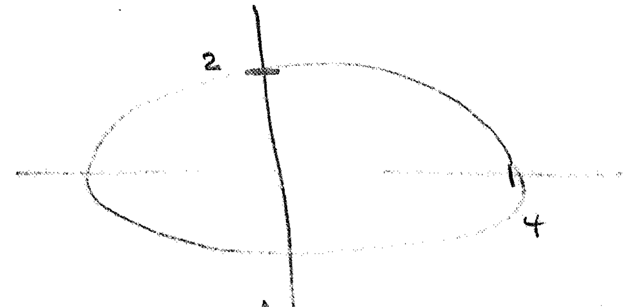
$$\frac{du}{dx} + 12u = -3. \quad \text{This is linear with int factor } e^{12x}$$

Thus $$e^{12x} u = -\int 3e^{12x}\,dx = -\frac{1}{4}e^{12x} + c$$

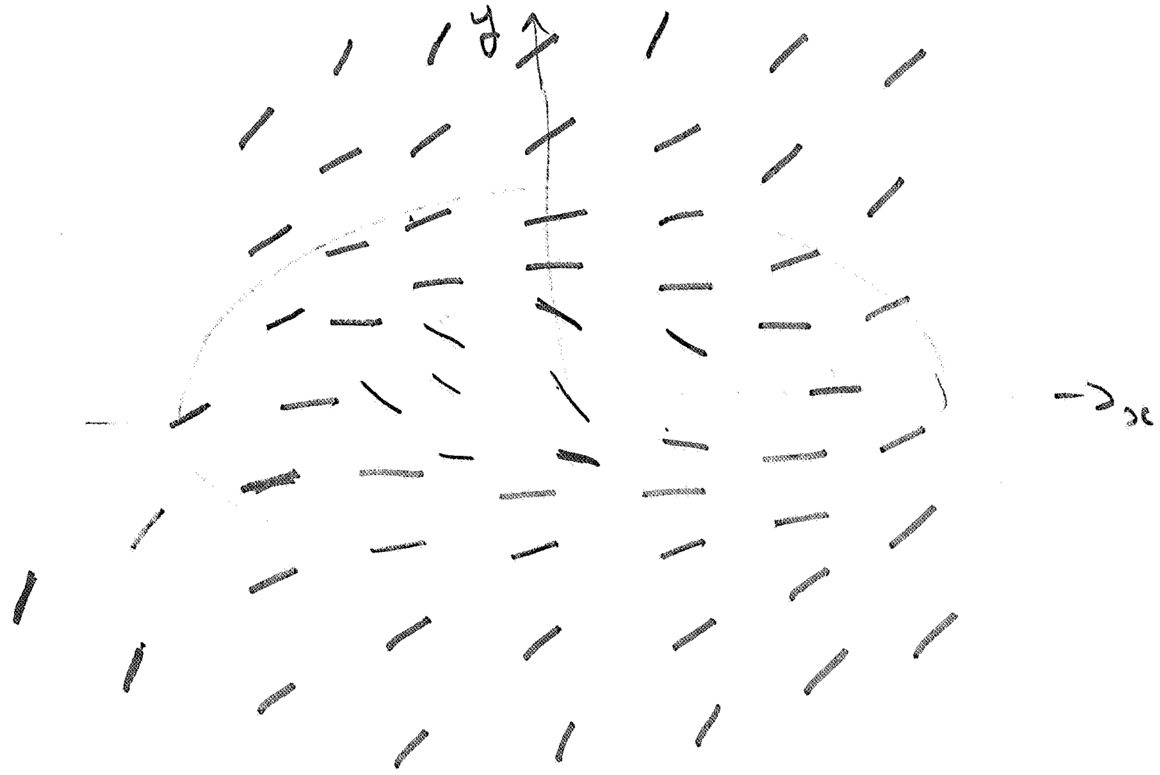
Hence $$u = y^{-3} = \left(ce^{-12x} - 1/4\right)$$

and $$y = \frac{1}{\left(ce^{-12x} - 1/4\right)^{1/3}}$$

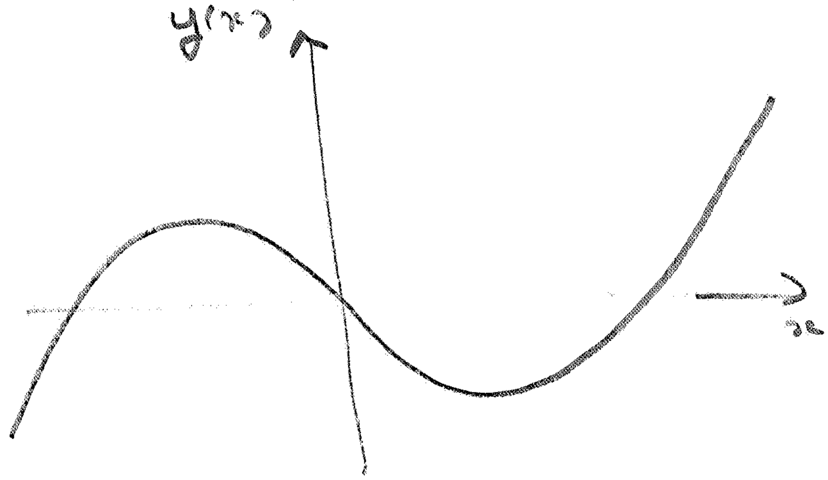
(b) Clearly $y' = 0$ when $\frac{x^2}{4} + y^2 = 4$ which is the ellipse.



Thus the direction field looks like.



The soln with $y(0) = 0$ will be of form.



2. (a) let $z = \frac{dy}{dx}$, then $\frac{dz}{dy} = \frac{dz}{dx} / \frac{dy}{dx} = \frac{y''}{y'}$

or $z \frac{dz}{dy} = y''$

The $z \frac{dz}{dy} = 6y^5$ and integrating.

$$z^2 = 2y^6 + c, \quad y'(0) = \sqrt{2}, \ y(0) = 1$$
$$\Rightarrow c = 0$$

and $z = \sqrt{2}\, y^3$. Thus $\int \frac{dy}{y^3} = \sqrt{2} \int dx$

and $-\frac{1}{2} \frac{1}{y^2} = \sqrt{2}\, x + c$

$y(0) = 1 \Rightarrow c = -\frac{1}{2}$

and so $-\frac{1}{2} \frac{1}{y^2} = \sqrt{2} x - \frac{1}{2} = -\frac{1}{2}(1 - 2\sqrt{2}\, x)$

or so $y^2 = \frac{1}{(1 - 2\sqrt{2}\, x)}$

& $y = \frac{1}{(1 - 2\sqrt{2}\, x)^{\frac{1}{2}}}$.

(b) The homo. eqn is $y'' - 3y' + 2y = 0$

with characteristic eqn $(\lambda^2 - 3\lambda + 2) = (\lambda - 2)(\lambda - 1) = 0$

Thus a FSS is $y^{(1)}(x) = e^{2x}$, $y^{(2)}(x) = e^{x}$.

A particular soln is.

$$y_p(x) = C_1(x)\, y^{(1)}(x) + C_2(x)\, y^{(2)}(x)$$

$$C_1(x) = -\int \frac{y^{(2)}(x) \cosh(x)\, dx}{W(y^{(1)}, y^{(2)})}$$

$$C_2(x) = +\int \frac{y^{(1)}(x) \cosh(x)\, dx}{W(y^{(1)}, y^{(2)})}.$$

The Wronskian is:

$$W(y^{(1)}(x), y^{(2)}(x)) = e^{2x} e^{x} (1-2) = -e^{3x}.$$

Thus $c_1(x) = +\int e^{-2x} \cosh(x)\,dx = \frac{1}{2}\int (e^{-x} + e^{-3x})\,dx$

$$= -\frac{1}{2}\left(e^{-x} + \frac{1}{3}e^{-3x}\right)$$

$c_2(x) = -\int e^{-x} \cosh(x)\,dx = -\frac{1}{2}\int (1 + e^{-2x})\,dx$

$$= -\frac{1}{2}\left(x - \frac{1}{2}e^{-2x}\right)$$

Thus $y_p(x) = -\frac{1}{2}\left(e^{x} + \frac{1}{3}e^{-x}\right) - \frac{1}{2}\left(xe^{x} - \frac{1}{2}e^{-x}\right)$

or $y_p(x) = -\frac{1}{2}xe^{x} + \left(\frac{1}{4} - \frac{1}{6}\right)e^{-x} = -\frac{1}{2}xe^{x} + \frac{1}{12}e^{-x}$

(since e^{x} satisfies the homo eqn).

[check $y_p' = -\frac{1}{2}e^{x}(1+x) - \frac{1}{12}e^{-x}$

$y_p'' = -\frac{1}{2}e^{x}(2+x) + \frac{1}{12}e^{-x}$

so $y_p'' - 3y_p' + 2y_p = -\frac{1}{2}e^{x}\left((2+x) - 3(1+x) + 2x\right)$

$+ \frac{1}{12}e^{-x}(1 + 3 + 2)$

$= (e^{x} + e^{-x})/2$ ✓]

Hence gen soln is:

$$y(x) = \frac{1}{12}e^{-x} - \frac{x}{2}e^{x} + Ae^{2x} + Be^{x}.$$

3. The eigenvalues are $2 \pm i$, with eigenvectors $\begin{pmatrix} 1 \mp i \\ 2 \end{pmatrix}$.

Thus there is a complex soln.

$$\begin{pmatrix} 1-i \\ 2 \end{pmatrix} e^{(2+i)t} = e^{2t}\begin{pmatrix} 1-i \\ 2 \end{pmatrix}(\cos(t) + i\sin(t))$$

$$= e^{2t}\begin{pmatrix} (\cos(t) + \sin(t)) \\ 2\cos(t) \end{pmatrix} + i e^{2t}\begin{pmatrix} \sin(t) - \cos(t) \\ 2\sin(t) \end{pmatrix}$$

Thus $\exists$ 2 linearly indep real solns.

$$e^{2t}\begin{pmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{pmatrix}, \quad e^{2t}\begin{pmatrix} \sin(t) - \cos(t) \\ 2\sin(t) \end{pmatrix}$$

& the gen soln is $\underline{x} = Ae^{2t}\begin{pmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{pmatrix}$

$$+ Be^{2t}\begin{pmatrix} \sin(t) - \cos(t) \\ 2\sin(t) \end{pmatrix}$$

$$\underline{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \end{pmatrix} + B\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so $2 = A - B$, $1 = 2A \Rightarrow A = \frac{1}{2}$, $B = A - 2 = -\frac{3}{2}$

& $$\underline{x}(t) = \frac{e^{2t}}{2}\left(\begin{pmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{pmatrix} - \begin{pmatrix} 3\sin(t) - 3\cos(t) \\ 6\sin(t) \end{pmatrix}\right)$$

$$= \frac{e^{2t}}{2}\begin{pmatrix} 4\cos(t) - 2\sin(t) \\ 2\cos(t) - 6\sin(t) \end{pmatrix} = e^{2t}\begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) - 3\sin(t) \end{pmatrix}$$

Thus

$$y(t) = \frac{3}{2} e^{-t} \sin(2t) + e^{-t}\left(\sin(t) - \frac{1}{2}\sin(2t)\right)$$

$$= e^{-t}\sin(2t) + e^{-t}\sin(t)$$

4. Taking Laplace trans gives.

$$\bar{y}(s)\,(s^2+2s+5) - sy(0) - y'(0) - 2y(0)$$
$$= \frac{3}{(s+1)^2+1}$$

s.t. $$\bar{y}(s)\,(s^2+2s+5) = 3\left(1 + \frac{1}{(s+1)^2+1}\right)$$

or $$\bar{y}(s) = \frac{3}{(s+1)^2+4}\left(1 + \frac{1}{(s+1)^2+1}\right)$$

Thus $$\bar{y}(s) = \frac{3}{2}\mathcal{L}\left[e^{-t}\sin(2t)\right] + \frac{3}{2}\mathcal{L}\left[e^{-t}\sin(2t)\right]\mathcal{L}\left[e^{-t}\sin(t)\right]$$

Hence $$y(t) = \frac{3}{2}e^{-t}\sin(2t) + \frac{3}{2}\int_0^t e^{-(t-\tau)}\sin(2(t-\tau))\,e^{-\tau}\sin(\tau)\,d\tau$$

latter integral $$= e^{-t}\int_0^t \sin(2(t-\tau))\sin(\tau)\,d\tau$$

$$= e^{-t}\int_0^t \frac{1}{2}\left(\cos(2t-3\tau) - \cos(2t-\tau)\right)d\tau$$

$$= \frac{e^{-t}}{2}\left[-\frac{\sin(2t-3\tau)}{3} + \sin(2t-\tau)\right]_0^t$$

$$= \frac{e^{-t}}{2}\left(-\frac{\sin(-t)}{3} + \frac{\sin(2t)}{3} + \sin(t) - \sin(2t)\right)$$

$$= \frac{e^{-t}}{2}\left(\frac{4}{3}\sin(t) - \frac{2}{3}\sin(2t)\right)$$

5. A FSS of the homo eqn is

$u_1(x) = \sin(x)$, $u_2(x) = \cos(x)$

linear combinations which obey the $x=0$, $x=\pi$ B.C. respectively are:

$y^{(1)}(x) = \sin(x)$, $y^{(2)}(x) = \cos(x)$

The Wronskian $W(y^{(1)}(s), y^{(2)}(s)) = -1$

Hence the Green's fn is:

$$G(x,s) = \begin{cases} -\sin(s)\cos(x) & 0 \le s < x \le \pi \\ -\sin(x)\cos(s) & 0 \le x < s \le \pi \end{cases}$$

Thus, the soln to eqn is:

$$y(x) = \int_0^{\pi} G(x,s)\, s\, ds$$

$$= -\int_0^{x} \sin(s)\cos(x)\, s\, ds - \int_x^{\pi} \sin(x)\cos(s)\, s\, ds.$$

$$= -\cos(x)\int_0^{x} \sin(s)\, s\, ds - \sin(x)\int_x^{\pi} \cos(s)\, s\, ds.$$

$$= -\cos(x)\left(-s\cos(s)\Big|_0^{x} + \int_0^{x} \cos(s)\, ds\right)$$

$$-\sin(x)\left(s\sin(s)\Big|_x^{\pi} - \int_x^{\pi} \sin(s)\, ds\right)$$

$$= -\cos(x)\left(-x\cos(x) + \sin(x)\right)$$
$$- \sin(x)\left(-x\sin(x) + (-1) - \cos(x)\right)$$

$$= +x\left(\cos^2(x) + \sin^2(x)\right)$$
$$+ \sin(x)\cos(x)(-1+1) + \sin(x)$$

$$= \sin(x) + x .$$

Thus $y(x) = \sin(x) + x$

6. Supposing $\underline{\lambda < 0}$, we may let $\lambda = -k^2$, $k > 0$ and the eqn becomes

$$y'' = k^2 y \text{ ; with gen soln}$$

$$y(x) = A\cosh(kx) + B\sinh(kx) .$$

$$2y(0) + y'(0) = 0 \iff 2A + kB = 0$$

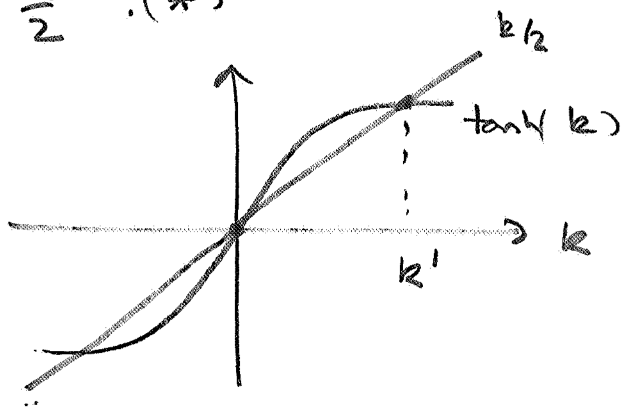
$$y(1) = 0 \iff A\cosh(k) + B\sinh(k) = 0 .$$

Thus for a non-zero soln, we require:

$$-\frac{k}{2}\cosh(k) + \sinh(k) = 0$$

$$\text{or} \quad \tanh(k) = \frac{k}{2} \quad .(*)$$

From the graph



we see there is one non-zero value k' that satisfies $(*)$. The corresponding eigenvalue and eigenfn are $\lambda = -k'^2$

and $\quad y(x) = -\frac{k}{2}\cosh(kx) + \sinh(kx)$.

<u>When $\lambda = 0$</u>, $y'' = 0$ & the gen soln is $y(x) = Ax + B$

$2y(0) + y'(1) = 0 \iff 2B + A = 0$

$y(1) = 0 \iff A + B = 0$

Thus $A = B = 0$ and there are no eigenvalues.

<u>When $\lambda > 0$</u>, let $\lambda = k^2 \; (k > 0)$, s.t.

$y'' = -k^2 y$ with gen soln

$y(x) = A\cos(kx) + B\sin(kx)$.

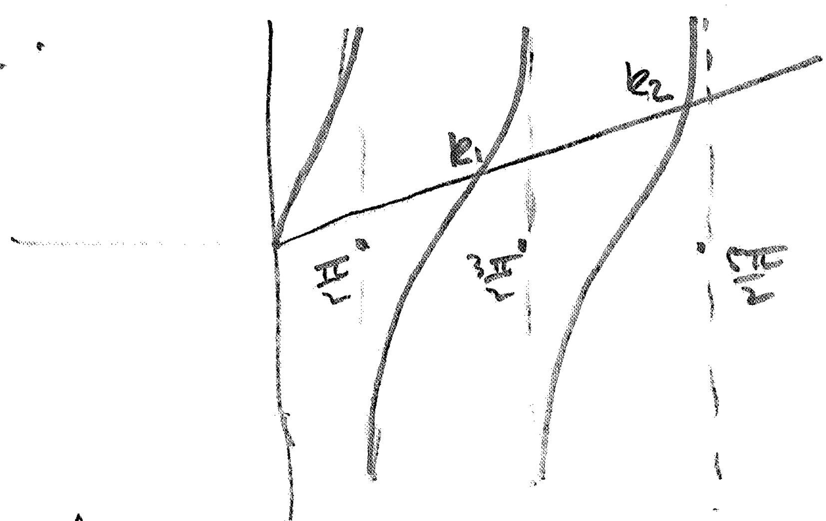
$2y(0) + y'(0) = 0 \iff 2A + kB = 0$

$y(1) = 0 \iff A\cos(k) + B\sin(k) = 0$.

We get non-zero solns if

$-\frac{k}{2}\cos(k) + \sin(k) = 0$, ie $\tan(k) = \frac{k}{2}$.

From the graph:



there are an infinite nor of solns $k_1, k_2, \ldots$.

$\frac{\pi}{2} < k_1 < \frac{3\pi}{2}$, $\frac{3\pi}{2} < k_2 < \frac{5\pi}{2}$, ...

The corresponding eigenvalues & eigenvectors are $\lambda = k_n^2$ and

$$y(x) = \left(-\frac{k_n}{2} \cos(k_n x) + \sin(k_n x) \right)$$

7. $\dot{x} = y$

$\dot{y} = x - x^2$

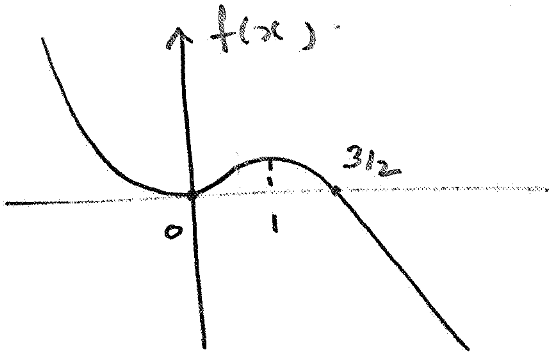
has equib point $(x, y) = (0,0), (1,0)$

Trajectories $y(x)$ obeys

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x - x^2}{y}$, thus $y^2 = x^2 - \frac{2}{3}x^3 + c$

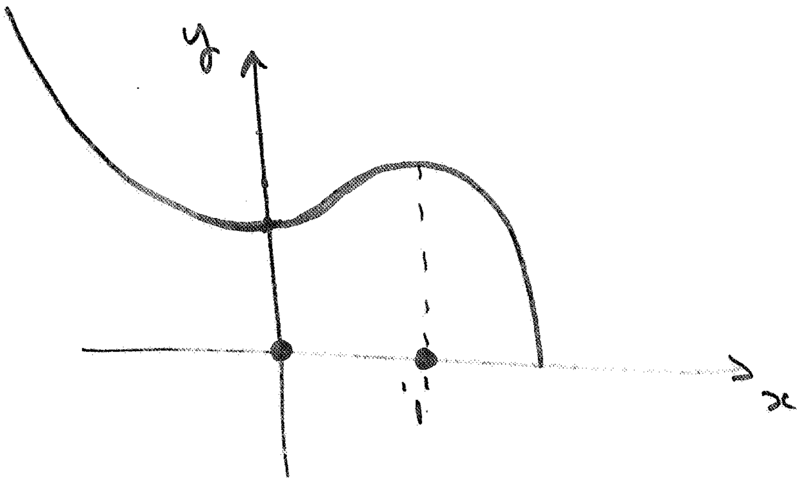
Consider the traj. through $(0, a>0)$. This will be $y^2 = \underbrace{x^2 - \frac{2}{3}x^3}_{f(x)} + a^2$

The fn $f(x)$ has $f'(x) = 2x - 2x^2 = 2x(1-x)$ and $f(x) \to \mp \infty$ as $x \to \pm \infty$. Thus we have

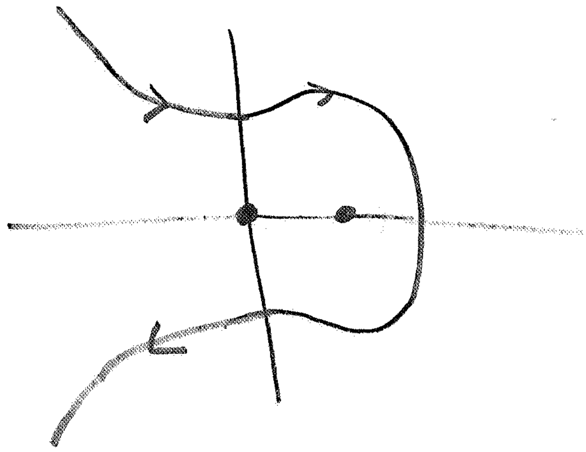


Hence, as x increases from 0, y increases until it reaches a max at $x = 1$, decreases until $y(3/2) = a$, and then decreases until $y = 0$. As x decrease from 0, y increases and $y \to \infty$ as $x \to -\infty$.

The traj. is of form.



By $y \to -y$ symmetry, we have the complete trajectory.



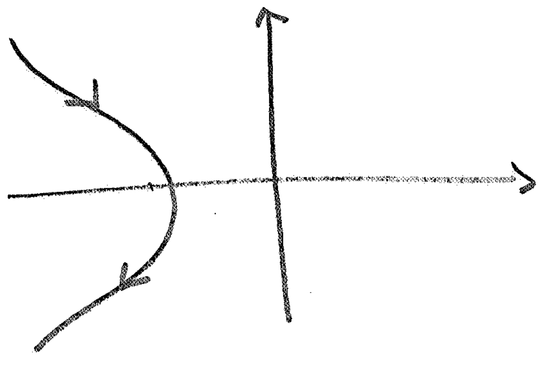
Consider the trajectory through $(b<0, 0)$.

This will of form.

$$y^2 = f(x) - f(b) \quad \text{with} \quad x < b \text{ for}$$

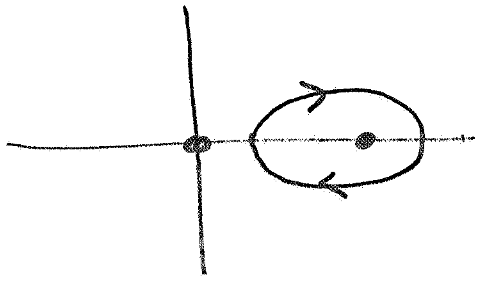
a real y.

As x decreases from b, y increase and we have.

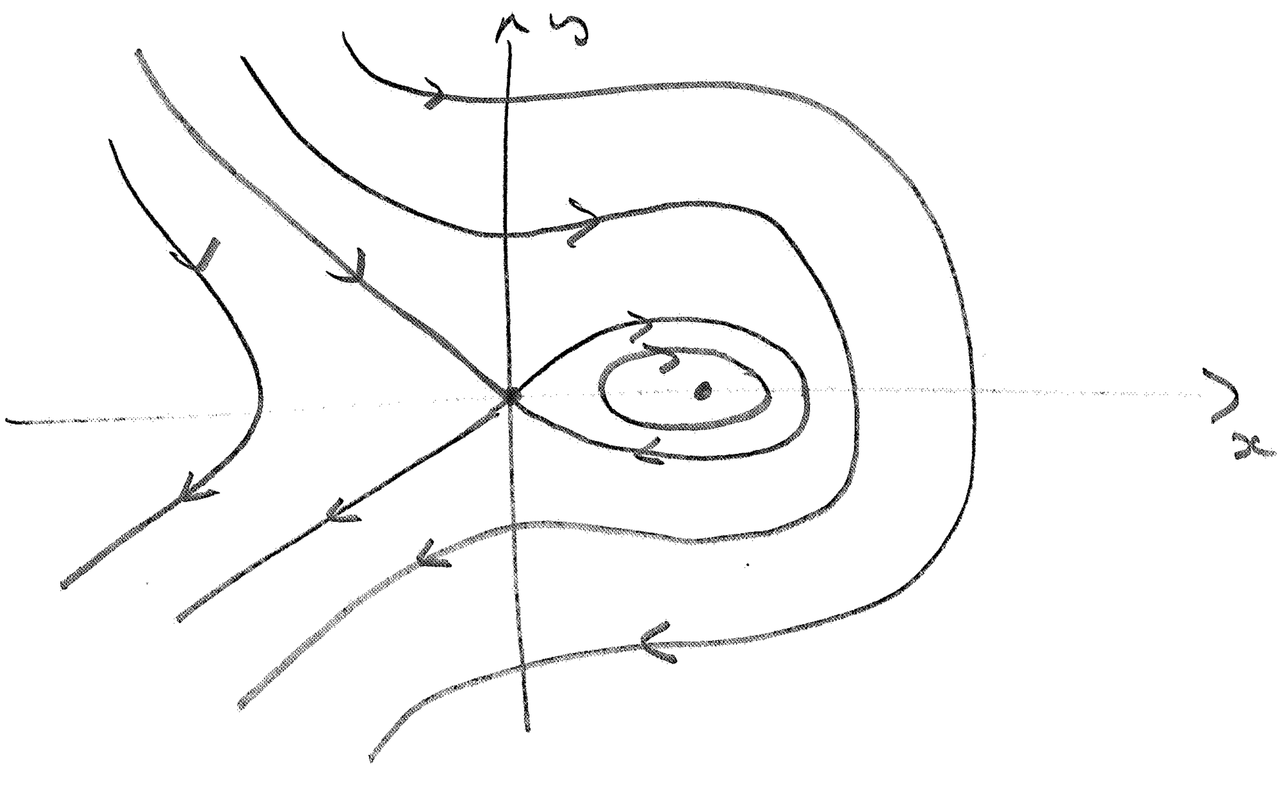


Consider the traj' through $(b, 0)$, $0 < b < 1$. This will be

$y^2 = f(x) - f(b)$, and will increase as x increases, reach a maximum at $x = 1$ and then decrease to zero at a value $1 < x < 3/2$. We have



The complete phase plane is of form.



8. (x,y) is an equib pt is

$$x(1-x-y) = 0 \quad ; \quad y(1+2x) = 0$$

So the equib pts are $(0,0)$, $(1,0)$, $(-\frac{1}{2}, 3/2)$

let $f(x,y) = x - xy - x^2$, $\quad g(x,y) = -y - 2xy$

Then the matrix of partial derivs is.

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-y-2x & -x \\ -2y & -1-2x \end{pmatrix}$$

The linearised eqn at $(0,0)$ is

$$\dot{\underline{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underline{x} .$$ This has eigenvals $1, -1$

and so $\underline{(0,0) \text{ is a saddle point.}}$

The linearised eqn at $(1,0)$ is

$$\dot{\underline{x}} = \begin{pmatrix} -1 & -1 \\ 0 & -3 \end{pmatrix} \underline{x}$$ with eigenvals $-1, -3$

so $(1,0)$ is a $\underline{\text{stable node}}$.

The linearised eqn at $(-1/2, 3/2)$ is

$$\dot{\underline{x}} = \begin{pmatrix} 1-3/2+1 & 1/2 \\ -3 & -1+1 \end{pmatrix} \underline{x} = \begin{pmatrix} 1/2 & 1/2 \\ -3 & 0 \end{pmatrix} \underline{x}$$

eigenvals obey

$$\left(\tfrac{1}{2}-\lambda\right)(-\lambda) + 3/2 = 0$$

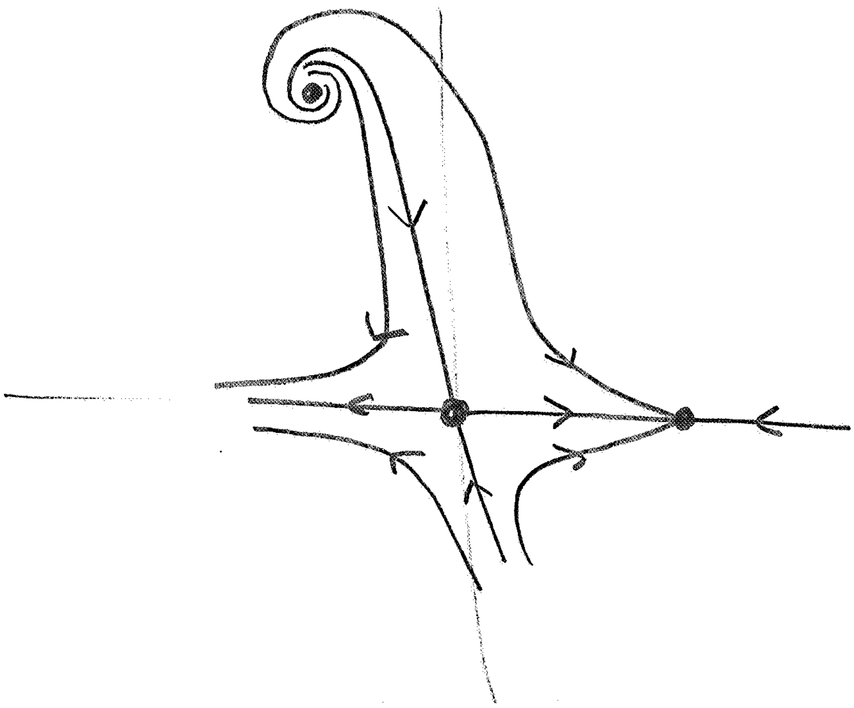
$$\lambda^2 - \tfrac{1}{2}\lambda + 3/2 = 0 \quad , \quad \text{or} \quad 2\lambda^2 - \lambda + 3 = 0$$

so $\lambda = \dfrac{1 \pm \sqrt{1-24}}{4} = 1/4 \pm \dfrac{i\sqrt{23}}{4}$

Thus $(-\tfrac{1}{2}, 3/2)$ is an unstable spiral point

(clockwise since $\dot{x} = \tfrac{1}{2}x + \tfrac{1}{2}y = -\tfrac{1}{4} + \tfrac{1}{2}y$ at $(-\tfrac{1}{2}, y)$)

A possible phase plane is .



9. letting $V(x,y) = ax^2 + by^2$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

$$= 2ax\,(x^3 - y^3) + 2by\,(2xy^2 + y)$$

$$= 2(ax^4 + by^2) + 2xy^3(-a + 2b)$$

If we choose $a = 2$, $b = 1$, then

$V(x,y) = 2x^2 + y^2$ has a minimum at

$(0,0)$ and

$$\frac{dV}{dt} = 4x^4 + 2y^2 \geq 0.$$

Hence $(0,0)$ is an unstable point.