Problem Sheet 13

Module F13YT2/YF3 **2006**

1. Determine if $(0,0)$ is a stable or unstable equilibrium point for each of the following systems

(i) $\dot{x} = -x^2;$ $\dot{y} = y^2$

(ii) $\dot{x} = -x^3;$ $\dot{y} = -y^3$

(iii) $\dot{x} = -y^3;$ $\dot{y} = x^3$

(iv) $\dot{x} = y^3;$ $\dot{y} = x^3.$

2. By constructing an appropriate Lyapunov function of the form $V(x,y) = ax^2 + by^2$, determine the stability of the equilibrium point at the origin for each of the following equations

(i) $\dfrac{dx}{dt} = -x^3 + xy^2;$ $\dfrac{dy}{dt} = -2x^2y - y^3$

(ii) $\dfrac{dx}{dt} = -\dfrac{1}{2}x^3 + 2xy^2;$ $\dfrac{dy}{dt} = -y^3$

(iii) $\dfrac{dx}{dt} = -x^3 + 2y^3;$ $\dfrac{dy}{dt} = -2xy^2$

(iv) $\dfrac{dx}{dt} = x^3 - y^3;$ $\dfrac{dy}{dt} = 2xy^2 + 4x^2y + 2y^3.$

3. Consider the system of equations

$$\frac{dx}{dt} = y - xf(x,y); \qquad \frac{dy}{dt} = -x - yf(x,y)$$

where f is a smooth function. Show, by considering the Lyapunov function $V(x,y) = x^2 + y^2$, that
(i) if $f(x,y) > 0$ in a neighbourhood of $(0,0)$, then $(0,0)$ is a stable equilibrium point;
(ii) if $f(x,y) < 0$ in a neighbourhood of $(0,0)$, then $(0,0)$ is an unstable equilibrium point.

4. Show that $V(x,y) = \alpha y + \beta x - a\ln(y) - b\ln(x)$ is a Lyapunov function for the predator prey system

$$\dot{x} = ax - \alpha xy; \qquad \dot{y} = -by + \beta xy$$

and deduce that the equilibrium point $(b/\beta, a/\alpha))$ is stable.