Prob Sheet 3: Linear systems of ODES 1

Module F13YT2

1. Find the general solution for each of the following systems of equations by first computing the eigenvalues and eigenvectors of an appropriate matrix.

$(i)\ \dot{x} = -3x + 2y; \quad \dot{y} = -x - y.$

$(ii)\ \dot{x} = 3x + 2y + 4z; \quad \dot{y} = 2x + 2z; \quad \dot{z} = 4x + 2y + 3z.$

$(iii)\ \dot{x} = -4x + y; \quad \dot{y} = -x - 2y.$

2. Show that all solutions of the system

$$\dot{x} = ax + by \qquad \dot{y} = cx + dy,$$

where a, b, c and d are constants, approach zero as $t \to \infty$ if $a + d < 0$ and $ad - bc > 0$.

3. Write the 3rd order linear homogenous ODE

$$\frac{d^3x}{dt^3} = a_1 x + a_2 \frac{dx}{dt} + a_3 \frac{d^2x}{dt^2}$$

as a system of 1st order ODEs of the form $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}$. Show that the condition that λ be an eigenvalue for A is just to the characteristic equation $\lambda^3 = a_1 + a_2\lambda + a_3\lambda^2$. Assuming that this equation has distinct real roots $\lambda_1, \lambda_2, \lambda_3$, write down the general solution of $\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}$.

4. Find the fundamental matrix $Y(t)$ for the linear system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}. \tag{1}$$

Compute $Y(t)^{-1}$ and hence solve the initial value problem given by (1) and the condition

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

5. Find the fundamental solution set of the system of equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2}$$

Hence, find the solution of the initial value problem consisting of (2) and the initial condition

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

6. Find the fundamental solution set of the system of equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$