Problem Sheet 7

Module F13YT2

1. Solve the initial value problem

$$y'' + 4y = 2t^3, \quad y(0) = 0, \;\; y'(0) = 0,$$

(i) Using Laplace transforms, making use of the convolution theorem.
(ii) Using Laplace transforms and partial fractions.
(iii) Using the method of undetermined coefficients.

Which method do you think is easiest? Those of a more adventurous persuasion can also try (iv) the method of variation of parameters (simple - but a bit messy in this case).

2. Another type of integral transform is called the 'Fourier transform'. The Fourier transform of a function $f(x)$ is defined to be $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$. Find the Fourier transform of the Dirac delta function $\delta(x-c)$ and of the function

$$\Delta_\varepsilon(x-c) = \begin{cases} 0 & \text{for } \; |x-c| > \varepsilon/2 \\ \frac{1}{\varepsilon} & \text{for } \; |x-c| \leq \varepsilon/2. \end{cases}$$

Find the limit $\varepsilon \to 0$ of $F[\Delta_\varepsilon(x-c)]$.

3. Find the function $y(t)$ (or generalised function) with Laplace transform $\bar{y}(s)$ given by

$$\frac{2}{s^2+1} + \frac{s\,e^{-s}}{s^2+2s+3} - e^{-3s}\frac{1}{s^4} + e^{-2s}.$$

4. Find the function $y(t)$ with Laplace transform $\bar{y}(s)$ given by

$$(i) \quad \bar{y}(s) = \frac{s}{s^2+5s+6}$$

$$(ii) \quad \bar{y}(s) = \frac{e^{-3s}2(s+2)}{s^2+2s+5}$$

5. Use the method of Laplace transforms to solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-2), \quad y(0) = 0, \;\; y'(0) = 1.$$

6. Use the method of Laplace transforms to solve the initial value problem

$$\frac{dx_1(t)}{dt} = x_1(t) + 2x_2(t); \quad \frac{dx_2(t)}{dt} = 2x_1(t) - 2x_2(t); \quad x_1(0) = 2, \quad x_2(0) = 1.$$