Problem Sheet 9

Module F13YT2 **2006**

1. Find the eigenvalues and eigenfunctions for the following boundary value problems:
$(i)\ -y'' = \lambda y; \quad y'(0) = 0, \ \ y'(1) = 0;$

$(ii)\ -y'' = \lambda y; \quad y(0) = 0, \ \ y'(1) = 0.$

2. Find the Fourier cosine series and the Fourier sine series for the function

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < 1 \\ 0 \text{ if } 1 \le x < 2. \end{cases}$$

3. Suppose $\alpha \in \mathbf{R}$ and consider the boundary value problem

$$-y'' = \lambda y, \qquad \alpha y(0) + y'(0) = 0, \quad y(1) = 0.$$

(i) Show that for all values of α there are infinitely many positive eigenvalues.
(ii) If $\alpha < 1$, show that there are no negative eigenvalues.
(iii) If $\alpha > 1$, show that there is a negative eigenvalue.
(iv) Show that $\lambda = 0$ is an eigenvalue if and only if $\alpha = 1$.

4. Find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n \phi_n$ of the function

$f(x) = x, \ 0 < x < 1$ using the eigenfunctions of (a) question $1\,(i)$ and (b) question $1\,(ii)$

5. Consider the boundary value problem

$$-y'' = \lambda y, \qquad y(0) = 0, \quad y(\pi) + y'(\pi) = 0.$$

Show that there exist infinitely many eigenvalues λ_n and find the corresponding eigenfunctions.
Show that an eigenfunction expansion for $f(x) = 1, \ 0 < x < \pi$ in terms of these eigenfunctions is given by $\sum_{n=1}^{\infty} a_n \sin(\sqrt{\lambda_n} x)$ where

$$a_n = \frac{2(1 - \cos(\sqrt{\lambda_n}\pi))}{\sqrt{\lambda_n}(\pi + \cos^2(\sqrt{\lambda_n}\pi))}.$$