Solutions 1: First Order Ordinary Differential Equations

Module F13YT2

1. (i) $y(x) = \frac{1}{6}\ln(3+6e^x) + C$ (ii) $y(x) = -x^2\cos x + 2x\sin x + 2\cos x + C$.

2. (i) $\dfrac{dy}{dx} = \dfrac{x^2}{y} \iff y\,dy = x^2\,dx \iff \dfrac{1}{2}y^2 = \dfrac{1}{3}x^3 + C \iff y^2 = \dfrac{2}{3}x^3 + K.$

(ii) $\dfrac{dy}{dx} + 3y = x + e^{-2x}$; Integrating factor $= \exp(\int 3\,dx) = e^{3x}$.
Equation may be rewritten as $\frac{d}{dx}(e^{3x}y) = xe^{3x} + e^x$.
Hence $e^{3x}y = \int xe^{3x} + \int e^x = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + e^x + C$ and so
$y = \frac{1}{3}x - \frac{1}{9} + e^{-2x} + Ce^{-3x}$.

(iii) Equation may be rewritten as $\dfrac{dy}{dx} + \dfrac{y}{x} = \cos(2x)$.
Integrating factor $= \exp(\int \frac{dx}{x}) = \exp(\ln x) = x$.
Equation may be rewritten as $\frac{d}{dx}(xy) = x\cos(2x)$.
Hence $xy = \int x\cos(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$ and so
$y = \frac{1}{2}\sin(2x) + \frac{1}{4x}\cos(2x) + \frac{C}{x}$.

(iv) $\dfrac{dy}{dx} = \dfrac{y^2+2xy}{x^2}$; equation is of homogeneous type.
Let $u = \frac{y}{x}$; then $y = ux$ and $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$.
Therefore $u + x\frac{du}{dx} = u^2 + 2u$ and so $\frac{du}{u^2+u} = \frac{dx}{x}$.
Hence $(\frac{1}{u} - \frac{1}{1+u})\,du = \frac{dx}{x}$ and so $\ln(\frac{u}{u+1}) = \ln(x) + C$.
Therefore $\frac{u}{u+1} = Kx$ and so $\frac{y}{x+y} = Kx$. Thus $y = \frac{Kx^2}{1-Kx}$.

(v) $xy^2 - x + (x^2y+y)\dfrac{dy}{dx} = 0$.
If we write the equation as $a(x,y)\dfrac{dy}{dx} + b(x,y) = 0$, then $\frac{\partial a}{\partial x} = 2xy = \frac{\partial b}{\partial y}$ and so the equation is exact.
Thus the equation has solution $\phi(x,y) = c$ where

$$\frac{\partial\phi}{\partial x} = xy^2 - x \quad (1) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = x^2y + y \quad (2)$$

(1) is satisfied if $\phi(x,y) = \frac{1}{2}x^2y^2 - \frac{1}{2}x^2 + f_1(y)$.
(2) is satisfied if $\phi(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + f_2(x)$.
Hence, choosing $f_1(y) = \frac{1}{2}y^2$ and $f_2(x) = -\frac{1}{2}x^2$, we obtain the solution
$\frac{1}{2}x^2y^2 - \frac{1}{2}x^2 + \frac{1}{2}y^2 = c$.

(vi) $(xe^y + 2y)\dfrac{dy}{dx} + e^y = 0$.
If we write the equation as $a(x,y)\dfrac{dy}{dx} + b(x,y) = 0$, then $\frac{\partial a}{\partial x} = e^y = \frac{\partial b}{\partial y}$ and so the equation is exact.
Thus the equation has solution $\phi(x,y) = c$ where

$$\frac{\partial\phi}{\partial x} = e^y \quad (1) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = xe^y + 2y \quad (2)$$

(1) is satisfied if $\phi(x, y) = xe^y + f_1(y)$.
(2) is satisfied if $\phi(x, y) = xe^y + y^2 + f_2(x)$.
Thus, choosing $f_1(y) = y^2$ and $f_2(x) = 0$, we obtain solution $xe^y + y^2 = c$.

(vii) is an exact eqn since

$$\frac{\partial}{\partial x}(\cos(x) + 2xe^{2y}) = -\sin(x) + 2e^{2y} = \frac{\partial}{\partial y}(1 - y\sin(x) + e^{2y}).$$

The equation can be written in the form

$$\frac{d\psi(x, y)}{dx} = 0, \quad \text{with soln } \psi(x, y) = C$$

where

$$\begin{aligned}\psi(x, y) &= \int(\cos(x) + 2xe^{2y})dy = y\cos(x) + xe^{2y} + f(x), \quad f(x) \text{ arbitrary},\\ &= \int(1 - y\sin(x) + e^{2y})dx = x + y\cos(x) + xe^{2y} + g(y), \quad g(y) \text{ arbitrary}.\end{aligned}$$

Compatibility requires

$$\psi(x, y) = x + y\cos(x) + xe^{2y},$$

and so the solution is given implicitly by

$$x + y\cos(x) + xe^{2y} = C.$$

$(viii)$ is of Bernoulli type. Let $u = y^{-2}$, and note that

$$\frac{du}{dt} = -2y^{-3}\frac{dy}{dt}.$$

Multiplying both side of (b) by $-2y^{-3}$ then gives

$$\frac{du}{dt} - 2u = -4.$$

This is linear; multiplying by an integrating factor e^{-2t} gives

$$\frac{d}{dt}(ue^{-2t}) = -4e^{-2t}.$$

Integrating gives

$$ue^{-2t} = 2e^{-2t} + c, \quad \text{such that } u = y^{-2} = (2 + c\,e^{2t})$$

Hence

$$y^2 = \frac{1}{2 + c\,e^{2t}}.$$

3. (i) $\frac{dy}{dx} = y^{1/2} \Longleftrightarrow y^{-1/2}dy = dx \Longleftrightarrow 2y^{1/2} = x + c \Longleftrightarrow y = \frac{1}{4}(x + c)^2$.
Since $y(0) = 0$, we require $c = 0$.
Noting that the above argument works only when $x > 0$, we obtain the solution $y(x) =$

$$\begin{cases} \frac{1}{4}x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Equation also has solution $y(x) \equiv 0$.

The hypotheses of Picard's Theorem are not satisfied as $y \to y^{1/2}$ does not have a continuous derivative at $y = 0$.

(ii) $\quad y\dfrac{dy}{dx} = x \Longleftrightarrow y\,dy = x\,dx \Longleftrightarrow \dfrac{1}{2}y^2 = \dfrac{1}{2}x^2 + c \Longleftrightarrow y^2 = x^2 + K.$

Since $y(0) = 0$, we require $K = 0$ and so $y^2 = x^2$, i.e., $y = \pm x$. Thus the equation has distinct solutions $y(x) = x$ and $y(x) = -x$.

The hypotheses of Picard's Theorem are not satisfied as the equation may be rewritten as $\dfrac{dy}{dx} = \dfrac{x}{y}$ and $y \to \frac{x}{y}$ is not a smooth function at $y = 0$.

4. Equation may be rewritten as $\dfrac{dy}{dx} = 1 - y^2$.

For constant solutions $1 - y^2 = 0$, i.e., $(1-y)(1+y) = 0$, i.e., $y = \pm 1$.

Hence constant solutions are $y(x) \equiv 1$ and $y(x) \equiv -1$.

If $y > 1$, $\dfrac{dy}{dx} < 0$; if $-1 < y < 1$, $\dfrac{dy}{dx} > 0$; if $y < -1$, then $\dfrac{dy}{dx} < 0$. For sketch of solutions see Maple print out.

5. $\dfrac{dy}{dx} = y^2 - x$.

$\dfrac{dy}{dx} = 0$ when $y^2 = x$; $\quad \dfrac{dy}{dx} > 0$ when $y^2 > x$; $\quad \dfrac{dy}{dx} < 0$ when $y^2 < x$. For sketch of solution see Maple print out.

6. Clearly the points satistfying $(i), (ii), (iii)$ are those for which $y = x$, $\quad y > x$, $\quad y < x$. The direction field, and solution satisfying the initial condition are