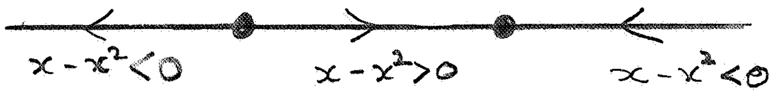
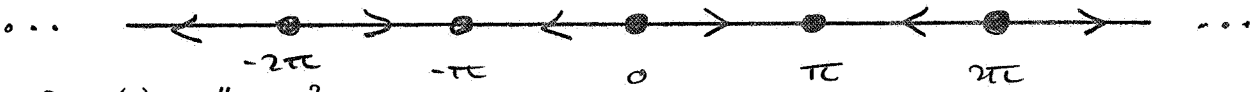
Solutions 10

Module F13YT2/YF3 **2006**

1. (i) $x' = x - x^2$.
Equilibrium points occur where $x - x^2 = 0$, i.e., $x = 0, 1$.



(ii) $x' = \sin(x)$.
Equilibrium points occur where $\sin(x) = 0$, i.e., $x = 0, \pm\pi, \pm 2\pi, \ldots$



2. (a) $x'' = -x^2$.

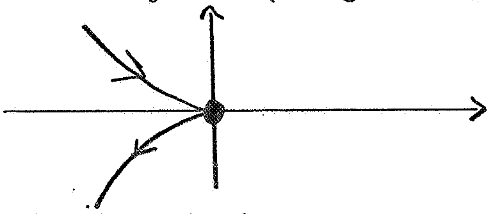
The equation may be written as the system $x' = y\,;\ y' = -x^2$.
Clearly $(0,0)$ is the only equilibrium point of the system.

Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = -\frac{x^2}{y}$, i.e., $y\,dy = -x^2\,dx$, i.e., $y^2 = -\frac{2}{3}x^3 + c$.

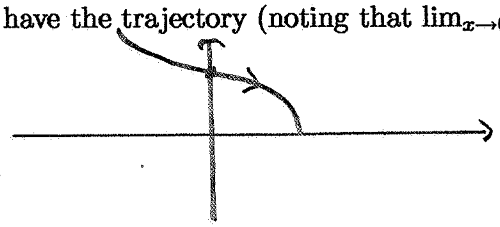
Let us consider the trajectories through some representative points.

The trajectory through the equilibrium point $(0,0)$ has the equation
$y^2 = -\frac{2}{3}x^3$, i.e., $y = \pm\sqrt{-\frac{2}{3}x^3}$ for $x < 0$.

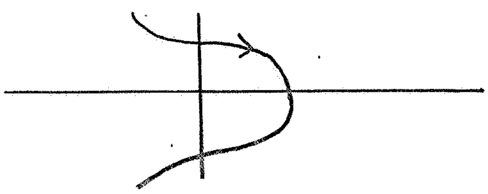
Hence we have the trajectories (noting that $\lim_{x\to 0} \dfrac{dy}{dx} = 0$).



Suppose $a > 0$ and consider the trajectory passing through $(0,a)$, i.e., $y^2 = -\frac{2}{3}x^3 + a^2$.
As x increases from 0, y decreases until it reaches 0 when $x = b := (3a^2/2)^{1/3}$. As x decreases from 0, y increases and $y \to \infty$ as $x \to -\infty$.
Hence we have the trajectory (noting that $\lim_{x\to 0} \dfrac{dy}{dx} = 0$, and $\dfrac{dy}{dx} \to -\infty$ as $y \to 0$).

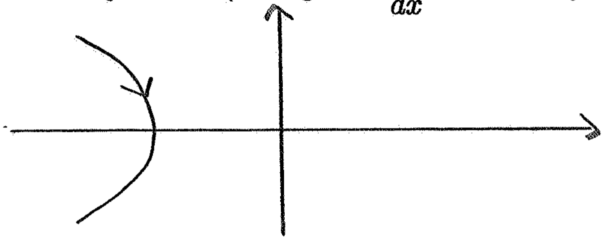


The trajectory is symmetric with respect to y and so we have

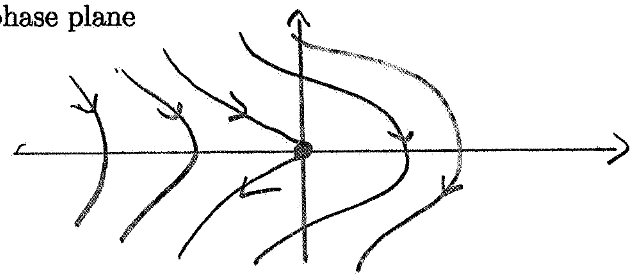


Clearly, we don't yet have enough trajectories to fill the phase plane, and we should consider a trajectory through the point $(b < 0, 0)$, i.e., $y^2 = -\frac{2}{3}x^3 + \frac{2}{3}b^3$. As x decreases from b, $y = \pm\sqrt{-\frac{2}{3}x^3 + \frac{2}{3}b^3}$, and as x increases their is no real y solution.

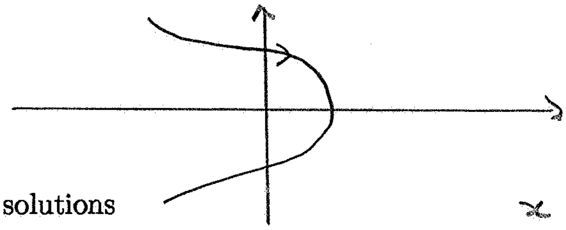
Hence, we have trajectories (noting that $\dfrac{dy}{dx} \to -\infty$ as $y \to 0$).



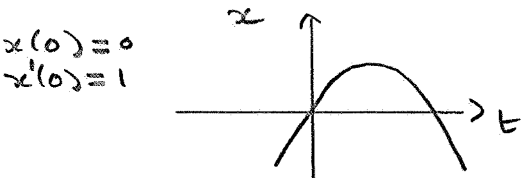
Hence we have the phase plane

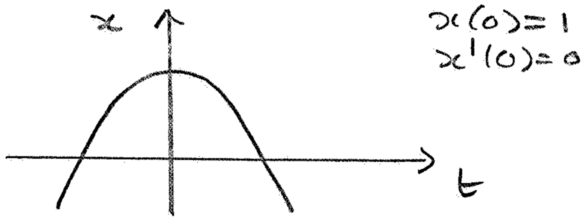


The points $(0, 1)$ and $(1, 0)$ both lie on a trajectory of the form



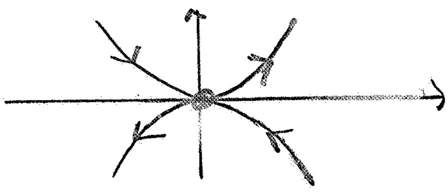
Hence we obtain solutions



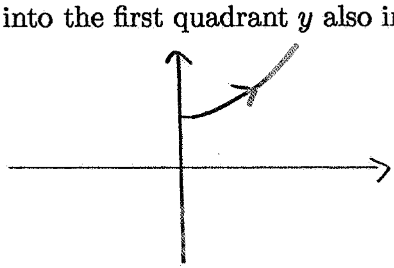


(ii) $x'' = x^3$.
The equation may be written as the system $x' = y$; $y' = x^3$.
Clearly $(0, 0)$ is the only equilibrium point of the system.
Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{x^3}{y}$, i.e., $y\,dy = x^3\,dx$, i.e., $y^2 = \frac{1}{2}x^4 + c$.

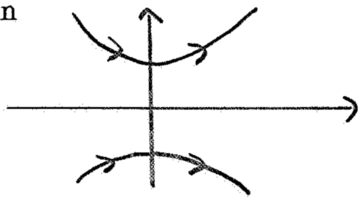
Trajectories approaching $(0, 0)$ have the equation $y^2 = \frac{1}{2}x^4$, i.e., $y = \pm\frac{1}{\sqrt{2}}x^2$.



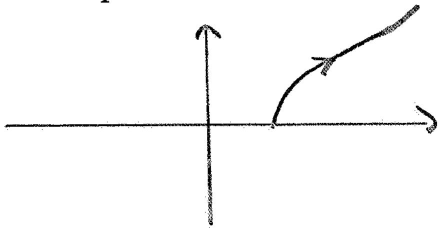
Consider the trajectory passing through $(0, a > 0)$, i.e., $y^2 = \frac{1}{2}x^4 + a^2$.
As x increases into the first quadrant y also increases and $y \to \infty$ as $x \to \infty$.



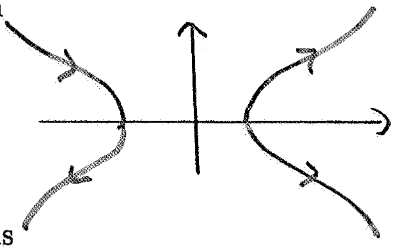
By symmetry we also obtain



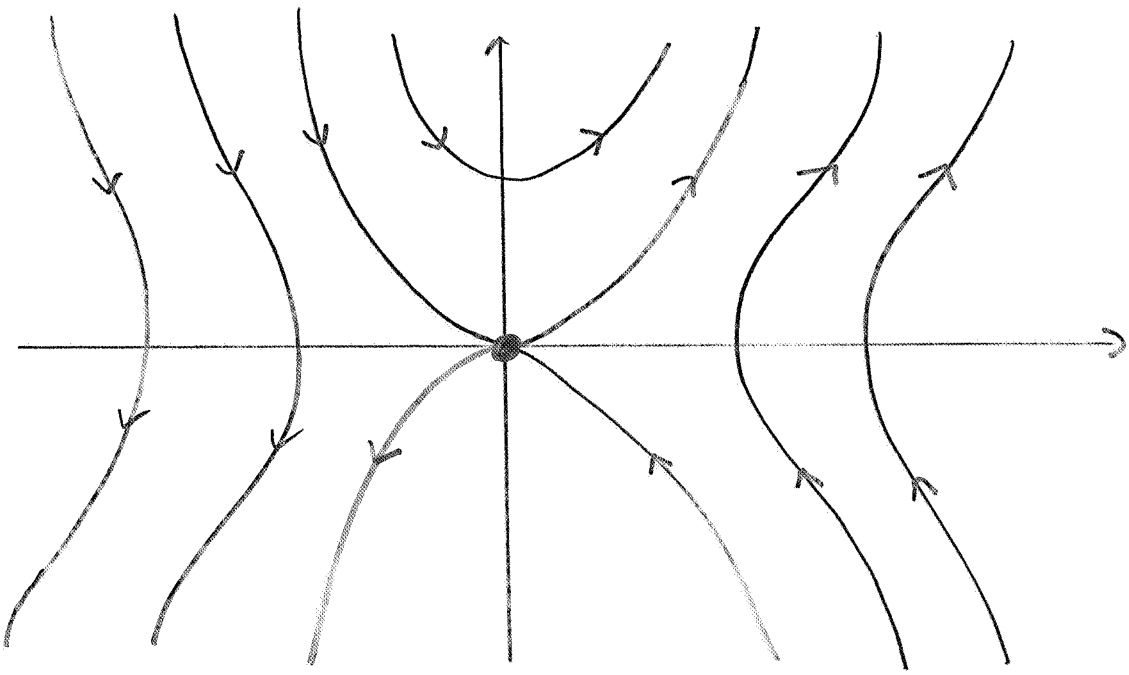
The trajectory passing through $(b > 0, 0)$ is $y^2 = \frac{1}{2}x^4 - \frac{1}{2}b^4$.
As y increases into the first quadrant x also increases and $x \to \infty$ as $y \to \infty$. There is no real solution y for $x < b$.



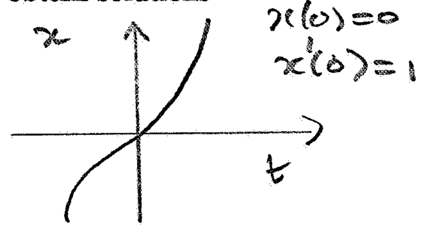
By symmetry we obtain

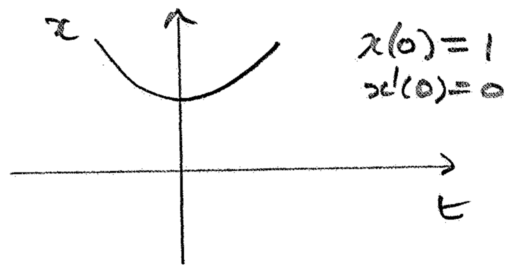


Hence the phase plane is



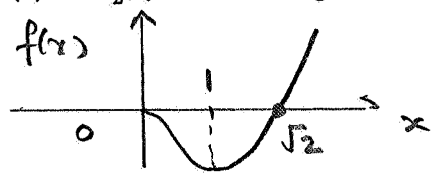
Hence we obtain solutions





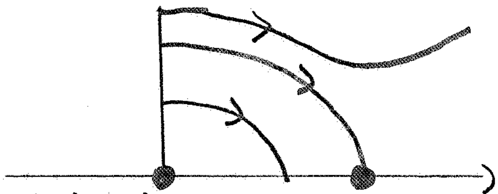
(iii) $x'' = x^3 - x$.
The equation may be written as the system $x' = y\,;\; y' = x^3 - x$.
Since $x^3 - x = 0$ if and only if $x = 0, 1, -1$, the equilibrium points of the system are $(0, 0)$, $(1, 0)$ and $(-1, 0)$.
Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{x^3 - x}{y}$, i.e., $y\,dy = (x^3 - x)\,dx$, i.e., $y^2 = \frac{1}{2}x^4 - x^2 + c$.

Let $f(x) = \frac{1}{2}x^4 - x^2$. Then $f'(x) = 2x^3 - 2x = 2x(x^2 - 1)$. Hence $f(0) = 0$, f is decreasing for $0 < x < 1$, $f(1) = -\frac{1}{2}$, f is increasing for $x > 1$ and $f(\sqrt{2}) = 0$. The graph is of the form

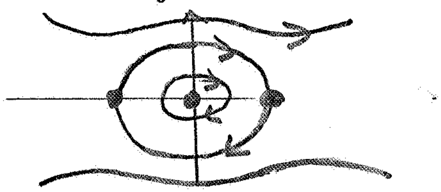


We may write the equation of trajectories as $y^2 = f(x) + c$.
Consider the trajectory passing through $(0, a > 0)$, i.e., $y^2 = f(x) + a^2$.
As x increases into the first quadrant, $f(x)$ and hence y are decreasing for $0 < x < 1$. The situation then depends upon whether y reaches 0 or not. There are 3 possibilities:
(i) If $a^2 + f(1) < 0$, then there exists $x_0 \in (0, 1)$ such that $f(x_0) = -a^2$ and $y(x_0) = 0$.
(ii) If $a^2 + f(1) = 0$, then $(1, 0)$ lies on the curve, i.e., the trajectory approaches the equilibrium point $(1, 0)$.
(iii) If $a^2 + f(1) > 0$, y is decreasing for $0 < x < 1$, $y > 0$ when $x = 1$. When x increases beyond 1, $f(x)$ and hence y are increasing.

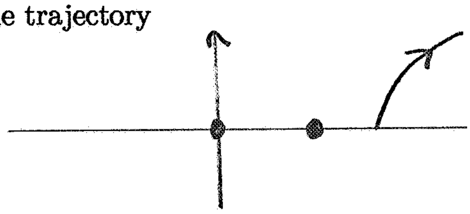
Hence we obtain the following trajectories



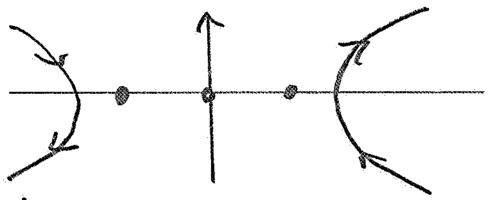
By symmetry we obtain the trajectories



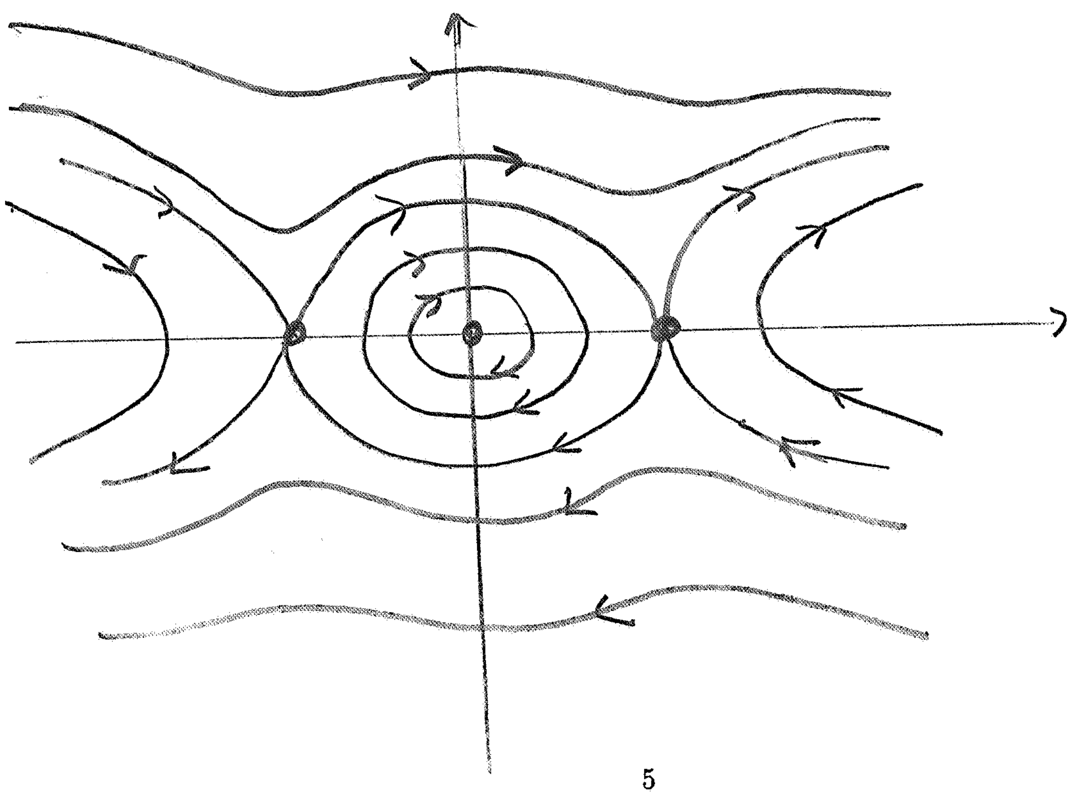
Consider the trajectory passing through $(a, 0)$ where $a \geq 1$,
i.e., $y^2 = f(x) - f(a)$.
As x increases from a, $f(x)$ and and y^2 increase. When $x < a$ there is no real y solution.
Thus we have the trajectory



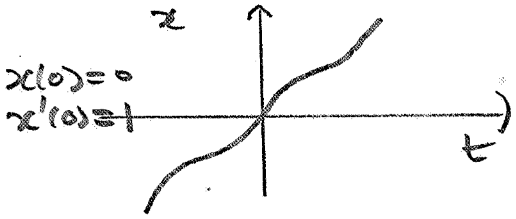
and by symmetry we obtain the trajectories

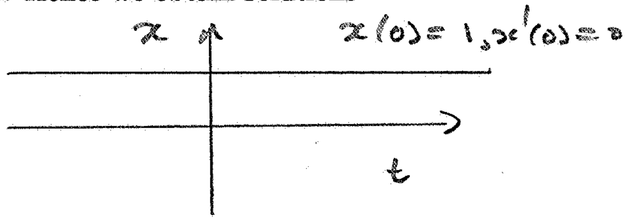


Hence phase plane is

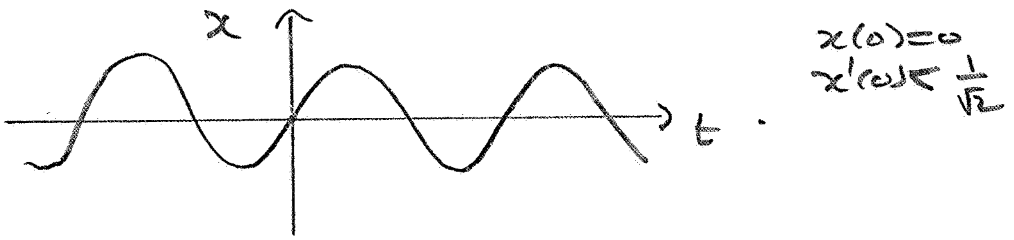


To consider the solutions note that $1^2 + f(1) = 1/2 > 0$. Hence we obtain solutions





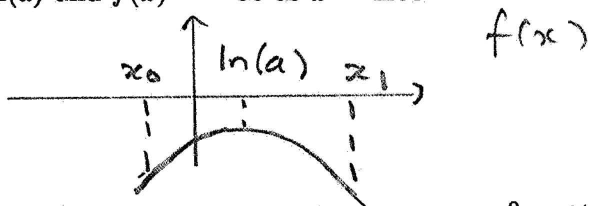
Note that $a^2 + f(1) \leq 0$ if and only if $a < \frac{1}{\sqrt{2}}$, in which case we would also have solutions like



3. $x'' + e^x = a$.

The equation may be written as the system $x' = y$; $y' = a - e^x$.
Supposing (i) $a > 0$, the system has an equilibrium point at $(\ln(a), 0)$.
Any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{a-e^x}{y}$, i.e., $y\,dy = (a - e^x)\,dx$, i.e., $y^2 = 2(ax - e^x) + c$.

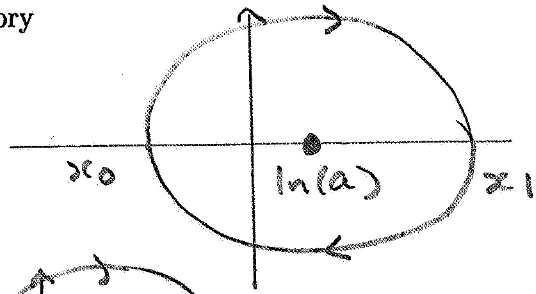
Let $f(x) = 2(ax - e^x)$. Then $f'(x) = 2(a - e^x)$. Hence f is increasing for $x < \ln(a)$ and decreasing for $x > \ln(a)$ and $f(x) \to -\infty$ as $x \to \pm\infty$.



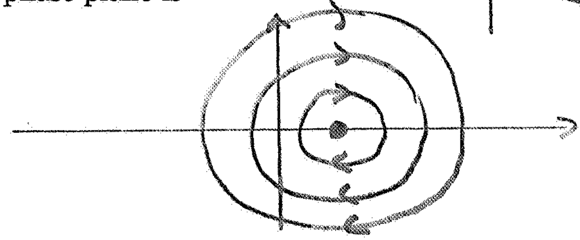
Clearly the equation of the trajectory can be written as $y^2 = f(x) + c$.

Consider the trajectory through $(x_0, 0)$, where $x_0 < \ln(a)$,
i.e., $y^2 = f(x) - f(x_0)$.
If $y^2 > 0$ on this trajectory, then $f(x) > f(x_0)$ and so we must have $x_0 < x < x_1$ where $x_1 > \ln(a)$ and $f(x_1) = f(x_0)$. As x increases from x_0 to $\ln(a)$, $f(x)$ increases and so y^2 increases; then as x increases from $\ln(a)$ to x_1, y^2 decreases to 0 again.

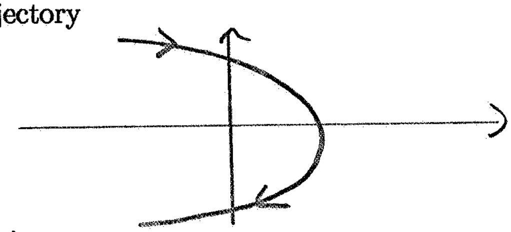
Thus we have the trajectory



and so phase plane is



If (ii) $a \leq 0$, then the system has no equilibrium points and $f(x) = 2(ax - e^x)$ is a strictly decreasing function such that $f(x) \to \mp\infty$ as $x \to \pm\infty$.
Consider the trajectory through $(x_0, 0)$, i.e., $y^2 = f(x) - f(x_0)$.
At all points on this trajectory we must have $f(x) \geq f(x_0)$ and so $x \leq x_0$.
Also as x decreases from x_0, y^2 increases and $y^2 \to \infty$ as $x \to -\infty$.
Hence we have the trajectory



and so the phase plane is