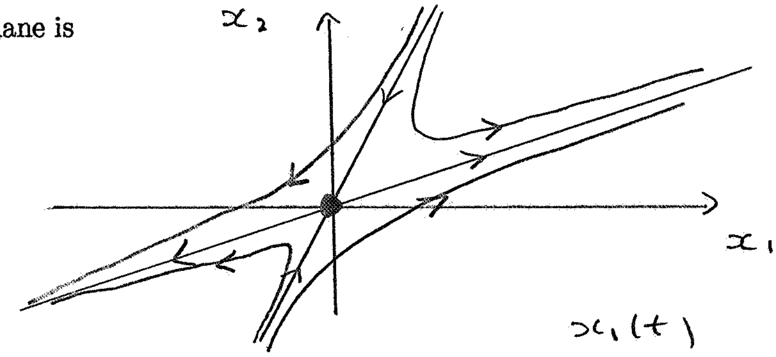
Solutions 11 - Module F13YT2/YF3

1.(a) $\dfrac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$

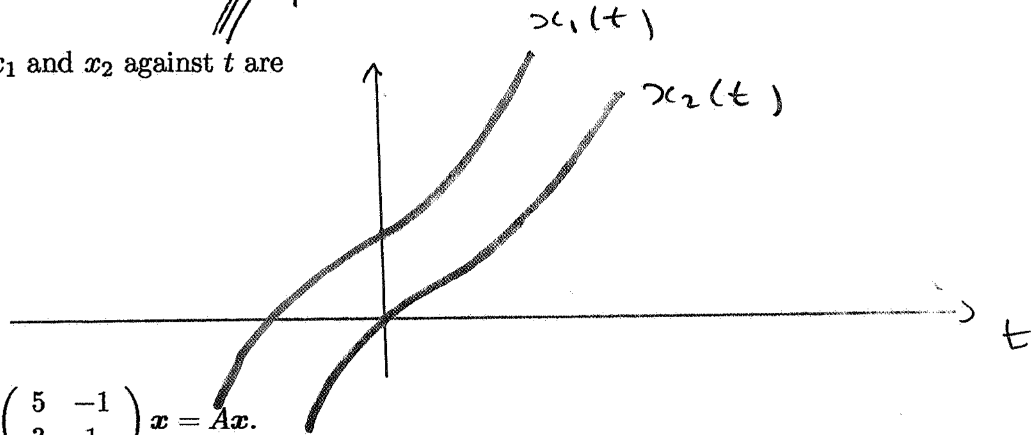
λ is an eigenvalue of A iff $\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0,$
i.e., $(\lambda-3)(\lambda+2)+4=0$, i.e., $\lambda^2-\lambda-2=0$,
i.e., $(\lambda-2)(\lambda+1)=0$, i.e., $\lambda=-1,2$.
Hence $(0,0)$ is a saddle point and so is unstable.

It is easy to check that the eigenvectors corresponding to $\lambda=-1$ and $\lambda=2$ are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Phase plane is



Graphs of x_1 and x_2 against t are

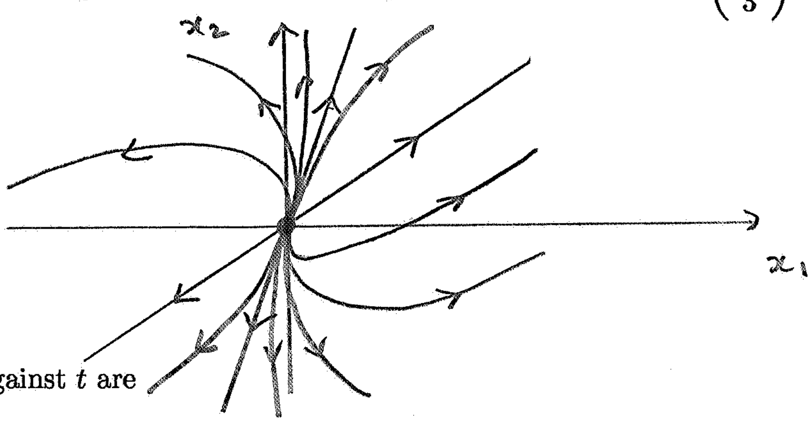


(b) $\dfrac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$

λ is an eigenvalue of A iff $\begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = 0,$
i.e., $(5-\lambda)(1-\lambda)+3=0$, i.e., $\lambda^2-6\lambda+8=0$,
i.e., $(\lambda-2)(\lambda-4)=0$, i.e., $\lambda=2,4$.
Hence $(0,0)$ is an unstable node.

It is easy to check that the eigenvectors corresponding to $\lambda=2$ and $\lambda=4$ are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Phase plane is



Graphs of x_1 and x_2 against t are

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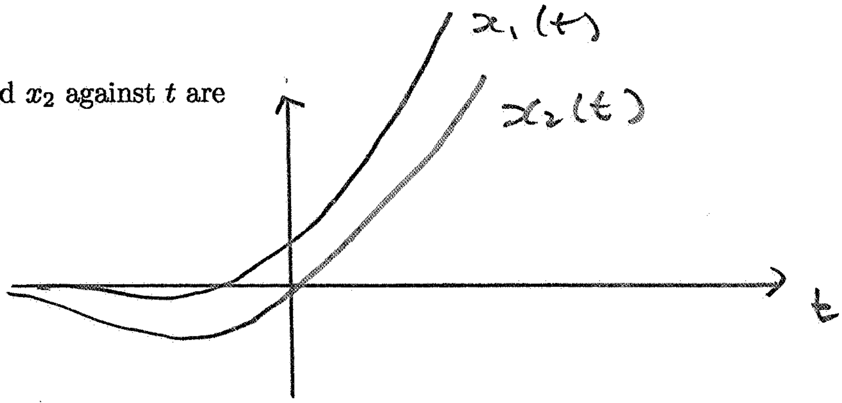


(c) $\dfrac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$

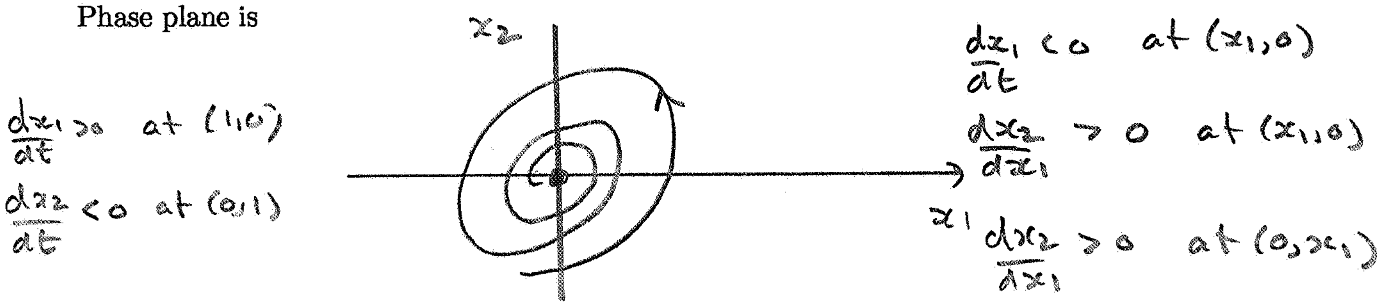
λ is an eigenvalue of A iff $\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0,$

i.e.,$(\lambda+3)(\lambda-1)+5=0$, i.e., $\lambda^2+2\lambda+2=0$,

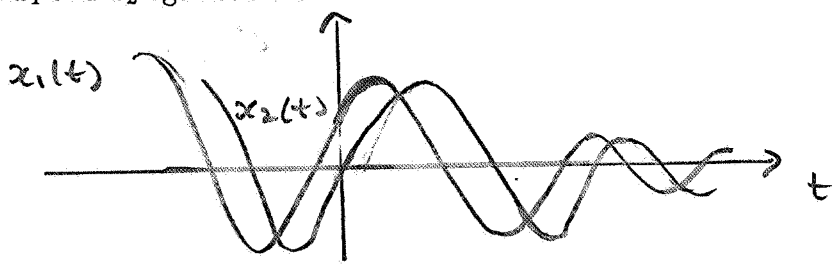
i.e., $\lambda = \frac{-2\pm\sqrt{-4}}{2}$, i.e., $\lambda = -1 \pm i$.

Hence $(0,0)$ is a stable spiral point and so is asymptotically stable.

Phase plane is



Graphs of x_1 and x_2 against t are



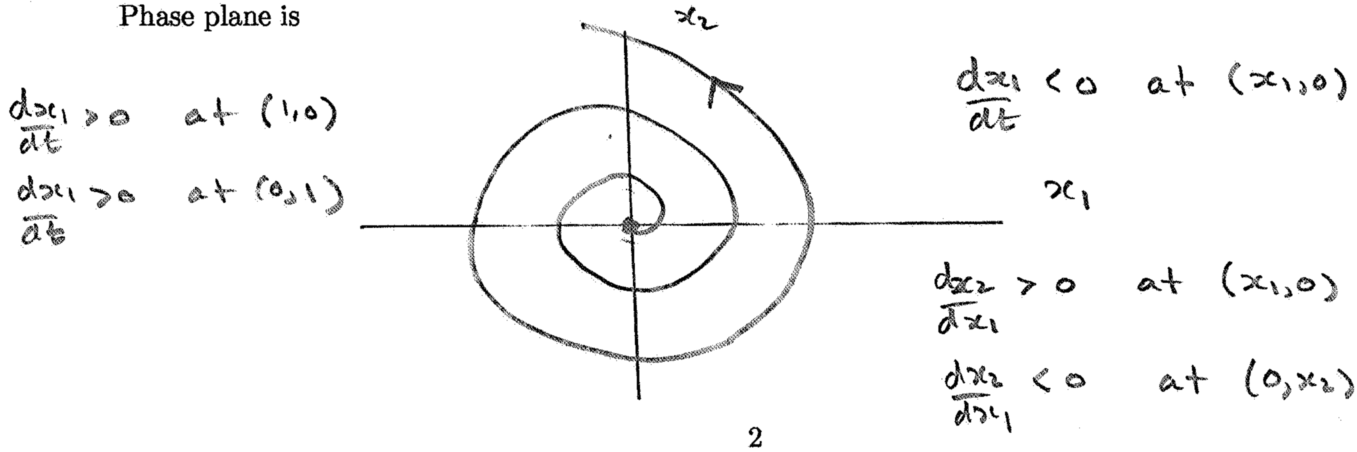
(d) $\dfrac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$

λ is an eigenvalue of A iff $\begin{vmatrix} 2-\lambda & -5 \\ 1 & 2-\lambda \end{vmatrix} = 0,$

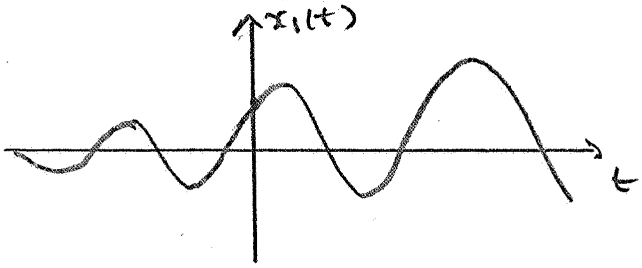
i.e.,$(2-\lambda)^2+5=0$, i.e., iff $2-\lambda = \pm\sqrt{5}i$, i.e., $\lambda = 2 \pm \sqrt{5}i$.

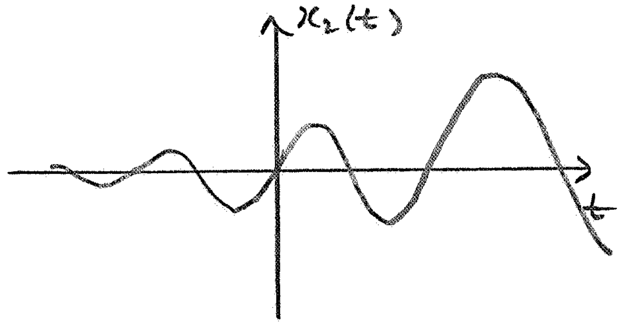
Hence $(0,0)$ is an unstable spiral point.

Phase plane is



Graphs of x_1 and x_2 against t are





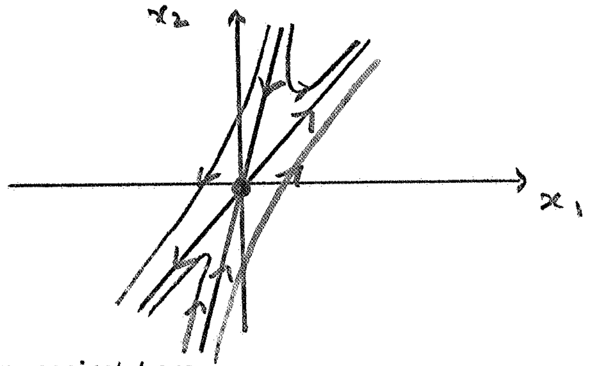
(e) $\dfrac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$

λ is an eigenvalue of A iff $\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0,$

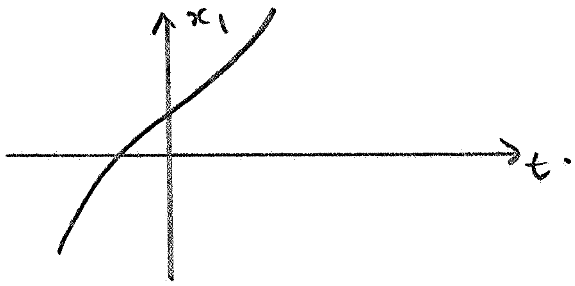
i.e., $(\lambda+2)(\lambda-2)+3=0$, i.e., iff $\lambda^2-1=0$, i.e., $\lambda = 1, -1$.
Hence $(0,0)$ is a saddle point and so is unstable.

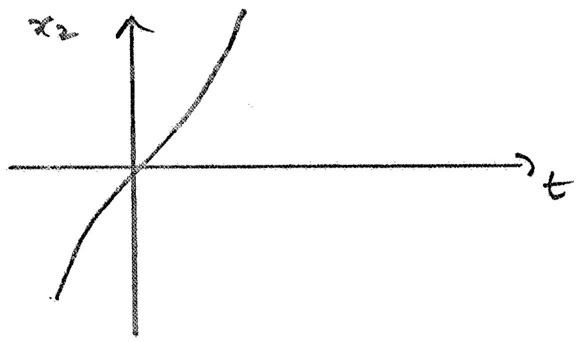
It is easy to check that the eigenvectors corresponding to $\lambda=-1$ and $\lambda=1$ are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Phase plane is



Graphs of x_1 and x_2 against t are





(f) $\dfrac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$

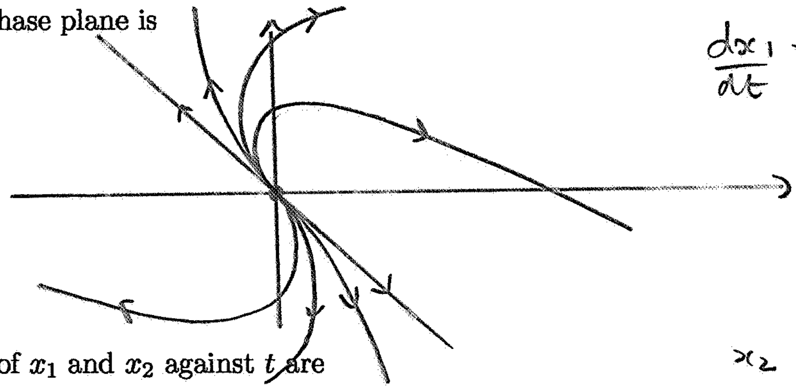
λ is an eigenvalue of A iff $\begin{vmatrix} 3-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = 0,$

i.e., $(\lambda+1)(\lambda-3)+4=0$, i.e., iff $\lambda^2-2\lambda+1=0$, i.e., iff $\lambda=1$.
Hence $(0,0)$ is an unstable ~~node~~. improper node .

It is easy to check that all eigenvectors corresponding to $\lambda=1$ are multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underline{\xi}$.

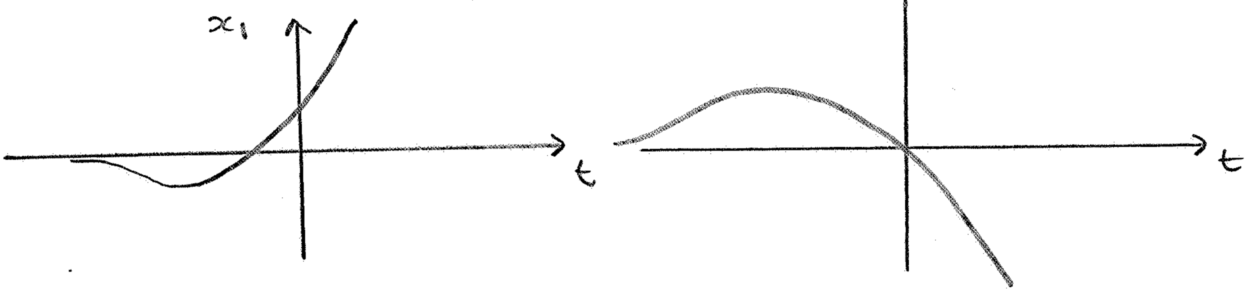
Hence system has one solution $e^t\,\underline{\xi}$ and a second solution of the form $te^t\underline{\xi}+e^t\underline{\eta}$ for some $\underline{\eta}$.
Thus general solution is $\boldsymbol{x}(t) = e^t\left[c_1\underline{\xi}+c_2t\,\underline{\xi}+c_2\underline{\eta}\right]$. Hence, as $t\to\infty$, $\boldsymbol{x}(t)$ approaches a multiple of $\underline{\xi}$ and, as $t\to-\infty$, $\boldsymbol{x}(t)$ approaches zero (again almost as a multiple of $\underline{\xi}$).

Hence phase plane is



$\frac{dx_1}{dt} > 0$ at $(0, x_2 > 0)$

Graphs of x_1 and x_2 against t are



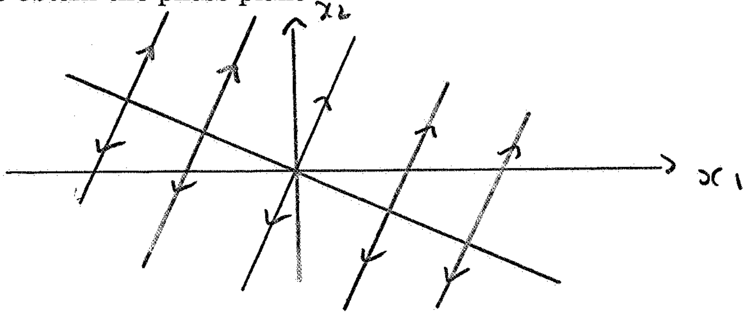
2. One eigenvalue is zero. We may write the system as $\frac{dx}{dt} = x + 2y$; $\frac{dy}{dt} = 2x + 4y$
Thus any trajectory of the form $y = y(x)$ must satisfy

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = -\frac{2x + 4y}{x + 2y} = 2$$

i.e., the trajectories are straight lines of the form $y = 2x + c$ where c is a constant.
Also (x, y) is an equilibrium point if and only if

$$x + 2y = 0; \qquad 2x + 4y = 0$$

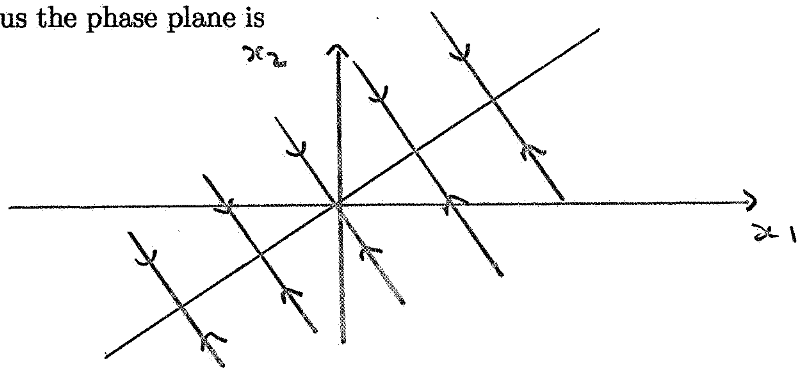
i.e., $x = -2y$. Thus the equilibrium points correspond to all points on the line $x + 2y = 0$.
Thus we obtain the phase plane



Clearly $\frac{dx}{dt} > 0$ whenever $x > -2y$ and so the trajectories are in the direction shown.

3. Let $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Since $A\boldsymbol{v}_1 = \mathbf{0}$ it follows that $c\boldsymbol{v}_1$ is an equilibrium point for all $c \in \mathbf{R}$.
The general solution of the system is $\boldsymbol{x}(t) = c_1\boldsymbol{v}_1 + c_2 e^{-t}\boldsymbol{v}_2$.
If $c_2 \neq 0$, this solution corresponds to the trajectory which is the straight line through the point corresponding to $c_1\boldsymbol{v}_1$ in the direction of $c_2\boldsymbol{v}_2$; as $t \to \infty$, the trajectory approaches $c_1\boldsymbol{v}_1$ and, as $t \to -\infty$, the trajectory goes off to infinity in the direction of $c_2\boldsymbol{v}_2$.

Thus the phase plane is



4. We may write the equation as $\dot{x} = y$; $\quad \dot{y} = -\frac{1}{m}\,[kx + ry]$,

i.e., $\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ \frac{k}{m} & -\frac{r}{m} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

λ is an eigenvalue of A iff $\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{r}{m} - \lambda \end{vmatrix} = 0$,

i.e., $\lambda(\lambda + \frac{r}{m}) + \frac{k}{m} = 0$, i.e., $m\lambda^2 + r\lambda + k = 0$, i.e., $\lambda = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}$.
Hence, if $r \geq 4mk$, A has two negative eigenvalues and so $(0,0)$ is a stable node.
If $r^2 < 4mk$, A has two complex eigenvalues and so $(0,0)$ is a stable spiral point.