Solutions 12

Module F13YT2/YF3 **2006**

1 (i) Corresponding linear system is $\frac{dx}{dt} = x - y$; $\frac{dy}{dt} = x + y$,
i.e., $\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.
λ is an eigenvalue of A iff $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$,
i.e., iff $(\lambda - 1)^2 + 1 = 0$, i.e., iff $\lambda - 1 = \pm i$, i.e., iff $\lambda = 1 \pm i$.
Hence $(0,0)$ is an unstable spiral point.

(ii) Corresponding linear system is $\frac{dx}{dt} = x$; $\frac{dy}{dt} = x - 2y$,
i.e., $\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.
λ is an eigenvalue of A iff $\begin{vmatrix} 1-\lambda & 0 \\ 1 & -2-\lambda \end{vmatrix} = 0$,
i.e., iff $(1-\lambda)(-2-\lambda) = 0$, i.e., iff $\lambda = 1, -2$.
Hence $(0,0)$ is a saddle point.

2 (i) $\frac{dx}{dt} = 1 - xy$; $\frac{dy}{dt} = (x-1)y$.
(x, y) is an equilibrium point iff

$$\begin{aligned} 1 - xy &= 0 \quad (1) \\ (x-1)y &= 0 \quad (2) \end{aligned}$$

(2) is satisfied iff $x = 1$ or $y = 0$.
If $y = 0$, then (1) cannot hold.
If $x = 1$, then (1) is satisfied when $y = 1$.
Hence system has unique equilibrium point $(1,1)$.
If $f(x,y) = 1 - xy$, $\frac{\partial f}{\partial x}(x,y) = -y$, $\frac{\partial f}{\partial y}(x,y) = -x$ and so $\frac{\partial f}{\partial x}(1,1) = -1$ and $\frac{\partial f}{\partial y}(1,1) = -1$.
If $g(x,y) = (x-1)y$, $\frac{\partial g}{\partial x}(x,y) = y$, $\frac{\partial g}{\partial y}(x,y) = x - 1$ and so $\frac{\partial g}{\partial x}(1,1) = 1$ and $\frac{\partial g}{\partial y}(1,1) = 0$. 2 (i)(ctd)
Hence we have linearized equation $\dot{\boldsymbol{x}}(t) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.
λ is an eigenvalue of A iff $\begin{vmatrix} -1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$,
i.e., iff $\lambda(\lambda+1) + 1 = 0$, i.e., iff $\lambda^2 + \lambda + 1 = 0$, i.e., iff $\lambda = \frac{-1 \pm \sqrt{3}i}{2}$.
Hence $(0,0)$ is a stable spiral point for the linearized equation and so $(1,1)$ is a stable spiral point for the original equation.

(ii) $\frac{dx}{dt} = x + y^2$; $\frac{dy}{dt} = x + y$.
(x, y) is an equilibrium point iff

$$\begin{aligned} x + y^2 &= 0 \quad (1) \\ x + y &= 0 \quad (2) \end{aligned}$$

(2) is satisfied iff $x = -y$ and then (1) is satisfied iff $x + x^2 = 0$,
i.e., iff $x = 0, -1$.
Hence system has equilibrium points $(0,0)$ and $(-1,1)$.
If $f(x,y) = x + y^2$, $\frac{\partial f}{\partial x}(x,y) = 1$ and $\frac{\partial f}{\partial y}(x,y) = 2y$.
If $g(x,y) = x + y$, $\frac{\partial g}{\partial x}(x,y) = 1$ and $\frac{\partial g}{\partial y}(x,y) = 1$.

Hence linearized equation at $(0,0)$ is $\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

Hence A has eigenvalues $\lambda = 1, 1$.

Thus $(0,0)$ is an unstable node of the linearized equation and so is either an unstable node or an unstable spiral point of the original equation.

Also linearized equation at $(-1,1)$ is $\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

λ is an eigenvalue of A iff $\begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = 0$,

i.e., iff $(\lambda - 1)^2 = 2$, i.e., iff $\lambda = 1 \pm \sqrt{2}$.

Hence $(0,0)$ is a saddle point for the linearized equation and so $(-1,1)$ is a saddle point for the original equation. 3. Equation can be written as the system $\dot{x} = y$; $y' = x^3 - x$.

(x,y) is an equilibrium point if $y = 0$ and $x^3 - x = 0$, i.e., $x = 0, \pm 1$.

Hence the equilibrium points are $(0,0)$, $(1,0)$ and $(-1,0)$.

If $f(x,y) = y$, $\frac{\partial f}{\partial x}(x,y) = 0$ and $\frac{\partial f}{\partial y}(x,y) = 1$.

If $g(x,y) = x^3 - x$, $\frac{\partial g}{\partial x}(x,y) = 3x^2 - 1$ and $\frac{\partial g}{\partial y}(x,y) = 0$.

Hence linearized equation at $(0,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

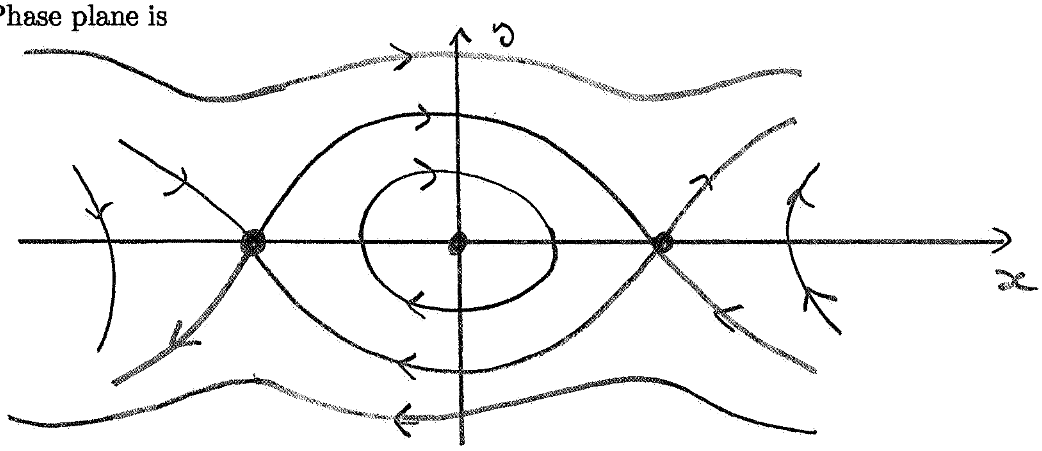
The eigenvalues of A are $\lambda = \pm i$ and so $(0,0)$ is a centre for the linearized equation; we cannot immediately conclude that $(0,0)$ is a centre for the original equation but in fact this is the case. (see tutorial sheet on phase planes question $2(iii)$).

The linearized equation at $(1,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

The eigenvalues of A are $\lambda = \pm\sqrt{2}$ and so $(1,0)$ is a saddle point.

An identical argument shows that $(-1,0)$ is also a saddle point.

Phase plane is



4. (a) $\frac{dx}{dt} = x(1 - x - y)$; $\frac{dy}{dt} = y(\frac{3}{2} - y - x)$.

(x,y) is an equilibrium point iff

$$\begin{aligned} x(1 - x - y) &= 0 \\ y(\tfrac{3}{2} - y - x) &= 0. \end{aligned}$$

Equilibrium points are $(0,0)$, $(0, \frac{3}{2})$, $(1,0)$. (Clearly it is impossible that both $1 - x - y = 0$ and $\frac{3}{2} - y - x = 0$ hold for the same values of x and y and so there are no equilibrium points with both $x \neq 0$ and $y \neq 0$).

If $f(x,y) = x(1 - x - y)$, $\frac{\partial f}{\partial x}(x,y) = 1 - 2x - y$ and $\frac{\partial f}{\partial y}(x,y) = -x$.

If $g(x,y) = y(\frac{3}{2} - y - x)$, $\frac{\partial g}{\partial x}(x,y) = -y$ and $\frac{\partial g}{\partial y}(x,y) = \frac{3}{2} - 2y - x$.

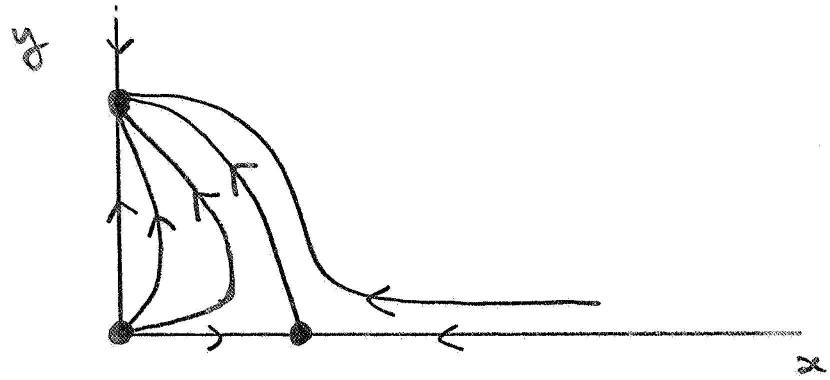
Hence linearized equation at $(0,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

Eigenvalues of A are $\lambda = 1, \frac{3}{2}$ and so $(0,0)$ is an unstable node. 4(a) (ctd) Linearized equation at $(0, \frac{3}{2})$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

Eigenvalues of A are $\lambda = -\frac{1}{2}, -\frac{3}{2}$ and so $(0, \frac{3}{2})$ is a stable node.

Linearized equation at $(1,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

Eigenvalues of A are $\lambda = -1, \frac{1}{2}$ and so $(1,0)$ is a saddle point.
Possible phase plane is



(b) $\frac{dx}{dt} = x(\frac{3}{2} - x - \frac{1}{2}y)$; $\quad \frac{dy}{dt} = y(2 - y - \frac{3}{4}x)$.
(x, y) is an equilibrium point iff

$$\begin{aligned} x(\tfrac{3}{2} - x - \tfrac{1}{2}y) &= 0 \\ y(2 - y - \tfrac{3}{4}x) &= 0 \end{aligned}.$$

Hence we have equilibrium points are $(0,0)$, $(0,2)$, $(\frac{3}{2}, 0)$ and (x, y) where

$$\begin{aligned} x + \tfrac{1}{2}y &= \tfrac{3}{2} \\ \tfrac{3}{4}x + y &= 2 \end{aligned},$$

i.e., where $(x, y) = (\frac{4}{5}, \frac{7}{5})$.
If $f(x,y) = x(\frac{3}{2} - x - \frac{1}{2}y)$, $\frac{\partial f}{\partial x}(x,y) = \frac{3}{2} - 2x - \frac{1}{2}y$ and $\frac{\partial f}{\partial y}(x,y) = -\frac{1}{2}x$.
If $g(x,y) = y(2 - y - \frac{3}{4}x)$, $\frac{\partial g}{\partial x}(x,y) = -\frac{3}{4}y$ and $\frac{\partial g}{\partial y}(x,y) = 2 - 2y - \frac{3}{4}x$.
Hence linearized equation at $(0,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

Eigenvalues of A are $\lambda = \frac{3}{2}, 2$ and so $(0,0)$ is an unstable node.

Linearized equation at $(0,2)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & -2 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

Eigenvalues of A are $\lambda = -2, \frac{1}{2}$ and so $(0,2)$ is a saddle point. 4(b) (ctd) Linearized equation at $(0, \frac{3}{2})$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{4} \\ 0 & \frac{7}{8} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

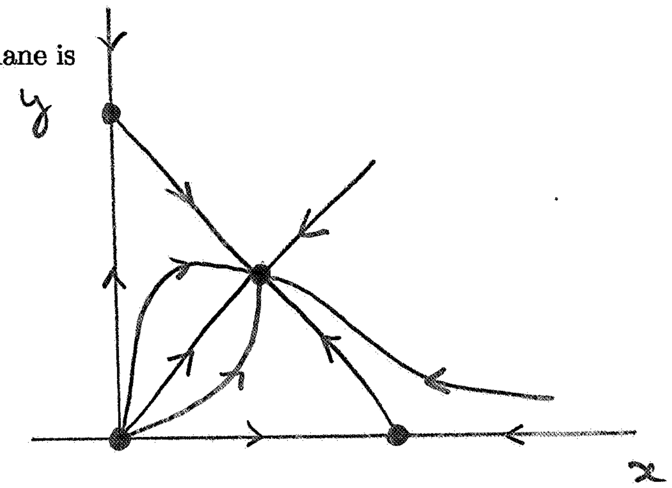
Eigenvalues of A are $\lambda = -\frac{3}{2}, \frac{7}{8}$ and so $(\frac{3}{2}, 0)$ is a saddle point.

Linearized equation at $(\frac{4}{5}, \frac{7}{5})$ is

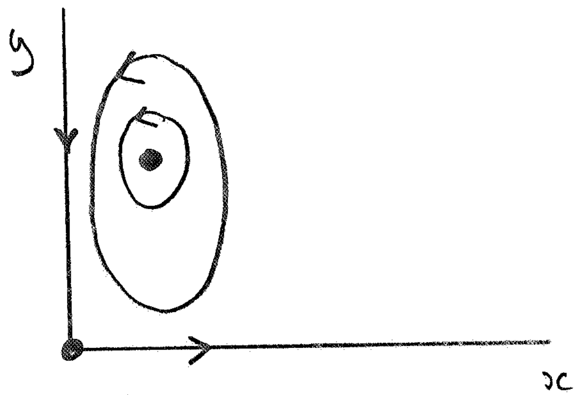
$$\dot{\boldsymbol{x}} = \begin{pmatrix} \frac{3}{2} - \frac{8}{5} - \frac{7}{10} & -\frac{2}{5} \\ -\frac{21}{20} & 2 - \frac{14}{5} - \frac{3}{5} \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} -\frac{4}{5} & -\frac{2}{5} \\ -\frac{21}{20} & -\frac{7}{5} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}.$$

λ is an eigenvalue of A iff $\begin{vmatrix} -\frac{4}{5} - \lambda & -\frac{2}{5} \\ -\frac{21}{20} & -\frac{7}{5} - \lambda \end{vmatrix} = 0$,
i.e., iff $(\lambda + \frac{4}{5})(\lambda + \frac{7}{5}) - \frac{21}{50}) = 0$, i.e., iff $\lambda^2 + \frac{11}{5}\lambda + \frac{7}{10} = 0$,
i.e., iff $\lambda = [-\frac{11}{5} \pm \sqrt{\frac{121}{25} - \frac{28}{10}}\,]/2 = \frac{-11 \pm \sqrt{51}}{10}$.
Hence both eigenvalues of A are negative and so $(\frac{4}{5}, \frac{7}{5})$ is a stable node.

Possible phase plane is



5. $\frac{dx}{dt} = x(\frac{3}{2} - \frac{1}{2}y)$; $\frac{dy}{dt} = y(-\frac{1}{2} + x)$.
Equilibrium points are $(0,0)$ and $(\frac{1}{2}, 3)$.
Since we have a standard predator prey system, $(0,0)$ is a saddle point and $(\frac{1}{2}, 3)$ is a centre and all trajectories in the first quadrant are closed curves centred at $(\frac{1}{2}, 3)$.
Hence the phase plane is



Equation of the trajectory $y = y(x)$ is given by

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{y(x - \frac{1}{2})}{x(\frac{3}{2} - \frac{1}{2}y)}.$$

Hence $(\frac{3}{2} . \frac{1}{y} - \frac{1}{2})\, dy = (1 - \frac{1}{2} . \frac{1}{x})\, dx$ and so
$\frac{3}{2}\ln(y) - \frac{1}{2}y - x + \frac{1}{2}\ln(x) =$ constant.

6. $\frac{dx}{dt} = x(1 - \frac{1}{2}x - \frac{1}{2}y)$; $\frac{dy}{dt} = y(-\frac{1}{4} + \frac{1}{2}x)$.
Equilibrium points are $(0,0)$, $(2,0)$, $(\frac{1}{2}, \frac{3}{2})$.

If $f(x,y) = x(1 - \frac{1}{2}x - \frac{1}{2}y)$, $\frac{\partial f}{\partial x}(x,y) = 1 - x - \frac{1}{2}y$ and $\frac{\partial f}{\partial y}(x,y) = -\frac{1}{2}x$.
If $g(x,y) = y(-\frac{1}{4} + \frac{1}{2}x)$, $\frac{\partial g}{\partial x}(x,y) = \frac{1}{2}y$ and $\frac{\partial g}{\partial y}(x,y) = -\frac{1}{4} + \frac{1}{2}x$.

(*i*) Linearized equation at $(0,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.
Eigenvalues of A are $\lambda = 1, -\frac{1}{4}$ and so $(0,0)$ is a saddle point.

(*ii*) Linearized equation at $(2,0)$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} -1 & -1 \\ 0 & \frac{3}{4} \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.
Eigenvalues of A are $\lambda = -1, \frac{3}{4}$ and so $(2,0)$ is a saddle point.

(*iii*) Linearized equation at $(\frac{1}{2}, \frac{3}{2})$ is $\dot{\boldsymbol{x}} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 \end{pmatrix} \boldsymbol{x} = A\boldsymbol{x}$.

λ is an eigenvalue of A iff $\begin{vmatrix} -\frac{1}{4}-\lambda & -\frac{1}{4} \\ \frac{3}{4} & -\lambda \end{vmatrix} = 0$,

i.e., iff $\lambda(\lambda+\frac{1}{4})+\frac{3}{16}=0$, i.e., iff $16\lambda^2+4\lambda+3=0$, i.e., iff $\lambda=\frac{-4\pm\sqrt{16-196}}{32}$.
Hence $(\frac{1}{2},\frac{3}{2})$ is a stable spiral and we have phase plane