Solutions 3

Module F13YT2

1. (i) $\dot{\boldsymbol{x}} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{x}$.

λ is an eigenvalue of A if and only if $\det\begin{bmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix} = 0$,

i.e., $\lambda^2 + 4\lambda + 3 + 2 = 0$, i.e., $\lambda = \frac{-4\pm\sqrt{16-20}}{2} = -2 \pm i$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2+i$ if and only if

$$\begin{aligned} -3x + 2y &= -2x + ix, \qquad \text{i.e., } (1+i)x = 2y \\ -x - y &= -2y + iy, \qquad \text{i.e., } x = (1-i)y, \qquad \text{i.e., } (1+i)x = 2y. \end{aligned}$$

Thus $\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2+i$.

Therefore the system has solution

$$\boldsymbol{x}(t) = e^{(-2+i)t}\begin{pmatrix} 2 \\ 1+i \end{pmatrix} = e^{-2t}(\cos(t) + i\sin(t))\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

i.e., system has solution $\boldsymbol{x}(t) = e^{-2t}\begin{pmatrix} 2\cos(t) + 2i\sin(t) \\ \cos(t) - \sin(t) + i(\cos(t) + \sin(t)) \end{pmatrix}$.

Taking real and imaginary parts we obtain the solutions

$e^{-2t}\begin{pmatrix} 2\cos(t) \\ \cos(t) - \sin(t) \end{pmatrix}$ and $e^{-2t}\begin{pmatrix} 2\sin(t) \\ \cos(t) + \sin(t) \end{pmatrix}$.

(ii) $\dot{\boldsymbol{x}} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\boldsymbol{x}$.

λ is an eigenvalue of A if and only if $\det\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} = 0$,

i.e., $(3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(-2-2\lambda) + 4(4+4\lambda) = 0$,
i.e., $\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$, i.e., $(\lambda+1)(\lambda^2 - 7\lambda - 8) = 0$,
i.e., $(\lambda+1)(\lambda+1)(\lambda-8) = 0$, i.e., $\lambda = -1, 8$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$ if and only if

$$\begin{aligned} 3x + 2y + 4z &= -x \\ 2x + 0y + 2z &= -y \qquad \text{i.e., if and only if } 2x + y + 2z = 0. \\ 4x + 2y + 3z &= -z \end{aligned}$$

Hence we have eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ corresponding to $\lambda = -1$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 8$ if and only if

$$\begin{aligned} 3x + 2y + 4z &= 8x \qquad \text{i.e.,} -5x + 2y + 4z = 0 \\ 2x + 0y + 2z &= 8y \qquad \text{i.e., } 2x - 8y + 2z = 0 \\ 4x + 2y + 3z &= 8z \qquad \text{i.e., } 4x + 2y - 5z = 0. \end{aligned}$$

Using Gaussian elimination we have the arrays

$$\begin{pmatrix} -5 & 2 & 4 \\ 1 & -4 & 1 \\ 4 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} -5 & 2 & 4 \\ 0 & -18 & 9 \\ 0 & 18 & -9 \end{pmatrix} \to \begin{pmatrix} -5 & 2 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e., $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an eigenvector if and only if $\begin{array}{rrrl} -5x & +2y & +4z & = 0 \\ & -2y & +z & = 0 \end{array}$

i.e., if $z = 2y$ and $5x = 2y + 4z = 10y$, i.e., $x = 2y$.

Hence $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 8$.

Thus the system has general solution

$$\boldsymbol{x}(t) = e^{-t}[c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}] + c_3\, e^{8t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

(iii) $\dot{\boldsymbol{x}} = \begin{pmatrix} -4 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{x}$.

λ is an eigenvalue of A if and only if $\det \begin{bmatrix} -4-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} = 0$,

i.e., $\lambda^2 + 6\lambda + 9 = 0$, i.e., $(\lambda + 3)^2 = 0$, i.e., $\lambda = -3$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -3$ if and only if

$$\begin{array}{ll} -4x + y = -3x & \text{i.e., } -x + y = 0. \\ -x - 2y = -3y & \text{i.e., } -x + y = 0. \end{array}$$

Thus $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -3$ if and only if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Then $\boldsymbol{x}(t) = e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution of the system.

We seek another solution of the form $\boldsymbol{x}(t) = te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}\mathbf{w}$.

Then we require $(e^{-3t} - 3te^{-3t}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3e^{-3t}\mathbf{w} = -3te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}A\mathbf{w}$,

i.e., $(A + 3I)\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Suppose $\mathbf{w} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then we require

$$\begin{array}{ll} -x + y = 1 & \\ -x + y = 1 & \text{e.g., } x = 0,\ y = 1 \end{array}$$

Thus we have second independent solution $\boldsymbol{x}(t) = te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence general solution is $\boldsymbol{x}(t) = e^{-3t}\,[c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t \\ t+1 \end{pmatrix}]$.

2. $\dot{\boldsymbol{x}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{x}$.

λ is an eigenvalue of A if and only if $\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$,

i.e., iff $(a - \lambda)(d - \lambda) - bc = 0$, i.e., iff $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$,

i.e., iff $\lambda = \frac{a+d\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$.
If $a+d<0$ and $ad-bc>0$, then either
(i) both eigenvalues are real and negative or
(ii) the eigenvalues are complex with negative real part.
Since the corresponding solutions have factors $e^{(\mathrm{Re}\lambda)t}$, it follows that all solutions $\to 0$ as $t\to\infty$.

3. We can write in this way with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

We have

$$(A-\lambda\mathbb{I}) = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a_1 & a_2 & a_3-\lambda \end{pmatrix}$$

and so $\det(A-\lambda\mathbb{I}) = a_1 + a_2\lambda + a_3\lambda^2 - \lambda^3$. Hence $\det(A-\lambda\mathbb{I}) = 0$ is equivalent to the characteristic equation given.
An eigenvector $\boldsymbol{v}$ corresponding to λ obeys

$$\begin{aligned} -\lambda v_1 + bv_2 &= 0 \\ -\lambda v_2 + v_3 &= 0 \\ a_1v_1 + a_2v_2 + (a_3-\lambda)v_3 &= 0. \end{aligned}$$

The 3rd equation follows from the other two and the characteristic equation. Hence we can choose

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}.$$

The general solution is then

$$\boldsymbol{x} = c_1e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{pmatrix} + c_2e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{pmatrix} + c_3e^{\lambda_3 t}\begin{pmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{pmatrix}.$$

4. The eigenvalues and eigenvectos of the amtrix are $-1, 2$ and $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. Hence a FSS is given by

$$e^{-t}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \quad e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The corresponding fundamental matrix is

$$Y(t) = \begin{pmatrix} \frac{1}{2}e^{-t} & 2e^{2t} \\ e^{-t} & e^{2t} \end{pmatrix}.$$

We find

$$Y(t)^{-1} = -\frac{2}{3}\begin{pmatrix} e^{t} & -2e^{t} \\ -e^{-2t} & \frac{1}{2}e^{-2t} \end{pmatrix}$$

Hence the solution is given by

$$\boldsymbol{x}(t) = Y(t)Y^{-1}(0)\begin{pmatrix}1\\2\end{pmatrix} = e^{-t}\begin{pmatrix}1\\2\end{pmatrix}.$$

Note that this isn't the easiest way of solving this equation!

5. There are 2 ways of solving this problem. First, the easier way:
If we denote the matrix by A, the eigenvalues are given by the roots of

$$\det(A - \lambda\mathbb{I}) = \lambda^2 - 4 = 0, \quad \text{such that} \quad \lambda = \pm 2.$$

The eigenvector $\boldsymbol{v}^{(1)}$ corresponding to $\lambda = 2$ is given by the soln of

$$\begin{aligned} v_1 + v_2 &= 2v_1 \\ 3v_1 - v_2 &= 2v_2 \end{aligned}$$

which reduce to the single equation $v_1 = v_2$. So we can choose $\boldsymbol{v}^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}$. The eigenvector $\boldsymbol{v}_2$ corresponding to $\lambda = 2$ is given by the soln of

$$\begin{aligned} v_1 + v_2 &= -2v_1 \\ 3v_1 - v_2 &= -2v_2 \end{aligned}$$

which reduce to the single equation $3v_1 = -v_2$. So we can choose $\boldsymbol{v}^{(2)} = \begin{pmatrix}1\\-3\end{pmatrix}$.
Hence, the general solution of (2) is

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = c_1e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + c_2e^{-2t}\begin{pmatrix}1\\-3.\end{pmatrix}$$

The solution of the initial value problem is obtained by choosing c_1 and c_2 to obey

$$\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

Solving, we find $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$, and so

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \frac{3}{4}e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{4}e^{-2t}\begin{pmatrix}1\\-3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3e^{2t} + e^{-2t}\\e^{2t} - 3e^{-2t}\end{pmatrix}.$$

The marginally more complicated way involves finding the eigenvalues and eigenvectors as above; computing the fundamental matrix

$$Y(t) = \begin{pmatrix}e^{2t} & e^{-2t}\\e^{2t} & -3e^{-2t}\end{pmatrix};$$

finding its inverse

$$Y(t)^{-1} = \frac{1}{4}\begin{pmatrix}3e^{-2t} & e^{-2t}\\e^{2t} & -e^{2t},\end{pmatrix}$$

and then using that

$$x(t) = Y(t)Y(0)^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{4}Y(t)\begin{pmatrix}3\\1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3e^{2t} + e^{-2t}\\e^{2t} - 3e^{-2t}\end{pmatrix}.$$

6. If we denote the matrix by A, the eigenvalues are given by the roots of

$$\det(A - \lambda\mathbb{I}) = \lambda^2 - 2\lambda + 5 = 0, \quad \text{such that} \quad \lambda = 1 \pm 2i.$$

The eigenvector $\boldsymbol{v}^{(1)}$ corresponding to $1 + 2i$ is given by the soln of

$$\begin{aligned} v_1 + 2v_2 &= (1 + 2i)v_1 \\ -2v_1 + v_2 &= (1 + 2i)v_2 \end{aligned}$$

which reduce to the single equation $v_2 = iv_1$. So we can choose $\boldsymbol{v}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. The eigenvector $\boldsymbol{v}^{(2)}$ corresponding to $\lambda = 1 - 2i$ is the complex conjugate $\boldsymbol{v}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. We can find two real solutions by taking the real and imaginary parts of the complex solution

$$\begin{aligned} e^{(1+2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} &= e^t(\cos(2t) + i\sin(2t)) \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t)) \end{pmatrix} + ie^t \begin{pmatrix} \sin(2t) \\ \cos(2t)) \end{pmatrix}. \end{aligned}$$

Thus we have two solutions

$$\boldsymbol{x}^{(1)}(t) = e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t)) \end{pmatrix}, \quad \boldsymbol{x}^{(2)}(t) = e^t \begin{pmatrix} \sin(2t) \\ \cos(2t)) \end{pmatrix}.$$