Solutions 4

Module F13YT2

1. The Wronskian is given by

$$W(y^{(1)}(t), y^{(2)}(t)) = y^{(1)}(t)\dot{y}^{(2)}(t) - \dot{y}^{(1)}(t)y^{(2)}(t) = \sin(2x)\cos(x) - 2\cos(2x)\sin(x).$$

2. The Wronskian is the determinant

$$\begin{vmatrix} e^t & e^{-t} & e^{2t} \\ e^t & -e^{-t} & 2e^{2t} \\ e^t & e^{-t} & 4e^{2t} \end{vmatrix} = e^{2t} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2t}$$

The solutions are linearly idependent because the Wronskian is non-zero at all t values.

3. The matrix has eigenvalues and eigenvectors $-1, 3$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Thus a fundamental matix is given by

$$Y(t) = \begin{pmatrix} -2e^{-t} & 2e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix}.$$

The inverse is

$$Y^{-1}(t) = \frac{1}{4}\begin{pmatrix} -e^{t} & 2e^{t} \\ e^{-3t} & 2e^{-3t} \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} \boldsymbol{x}(t) &= Y(t)\int_0^t Y^{-1}(\tau)e^{\tau}\begin{pmatrix} 1 \\ 0 \end{pmatrix} d\tau \\ &= \frac{1}{4}Y(t)\int_0^t \begin{pmatrix} -e^{2\tau} \\ e^{-2\tau} \end{pmatrix} d\tau \\ &= -\frac{1}{8}Y(t)\begin{pmatrix} e^{2t}-1 \\ e^{-2t}-1 \end{pmatrix} \end{aligned}$$

We can simplify before working out by noting that since $\frac{1}{8}Y(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution of the homegenous equation we can subtract it from the particular solution to obtain another particular solution

$$\boldsymbol{x}(t) = -\frac{1}{8}Y(t)\begin{pmatrix} e^{2t} \\ e^{-2t} \end{pmatrix} = -\frac{1}{4}e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

4. Two solutions of the homogenous eqn are $y^{(1)}(x) = \cos(wx)$ and $y^{(2)} = \sin(wx)$. The corresponding Wronskian is

$$W(y^{(1)}(x), y^{(2)}(x)) = w.$$

Hence a particular solution is given by

$$y(x) = \frac{1}{w}\int_0^x (y^{(1)}(z)y^{(2)}(x) - y^{(1)}(x)y^{(2)}(z))f(z)dz = \frac{1}{w}\int_0^x \sin(w(x-z))dz.$$

5. The characteristic equation for the homogeneous equation is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. Thus, a FSS is given by $y^{(1)}(x) = e^x$ and $y^{(2)}(x) = xe^x$. The Wronskian is

$$W(y^{(1)}(x), y^{(2)}(x)) = e^{2x}.$$

Hence a particular solution is given by

$$\begin{aligned} y_p(x) &= \int_{x_0}^{x} e^{-2z}(y^{(1)}(z)y^{(2)}(x) - y^{(1)}(x)y^{(2)}(z))\frac{e^z}{z}dz \\ &= \int_{x_0}^{x} (\frac{x}{z}e^x - e^x)dz = [xe^x \ln(z) - e^x z]_{x_0}^{x} \\ &= xe^x \ln(x) - e^x x - xe^x \ln(x_0) + e^x x_0. \end{aligned}$$

All terms but one are complementary functions (i.e. solutions of the homogenous equations). We can remove them and choose a particular solution

$$y_p(x) = xe^x \ln(x).$$

[**Note, it's always a good idea to substitute back and check that your particular solution really does satisfy the inhomgeneous equation**]
The general solution is then

$$y(x) = xe^x \ln(x) + Ae^x + Bxe^x.$$

6. (i) The char. eqn. is $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$ with roots $\lambda = 1, -2$. Thus, the gen soln is $x(t) = c_1e^t + c_2e^{-2t}$.

(ii) The char. eqn. is $\lambda^2 + 2\lambda + 2 = (\lambda + 1 - i)(\lambda + 1 + i) = 0$ with roots $\lambda = -1 \pm i$. Thus, we can obtain 2 real solution form $\text{Re}(e^{(-1+i)t}) = e^{-t}\cos(t)$ and $\text{Im}(e^{(-1+i)t}) = e^{-t}\sin(t)$. The gen soln is $x(t) = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$.

(iii) The char. polynomial is $\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda + 1)(\lambda - 1)(\lambda - 2) = 0$ (obtain by spotting one root, say $\lambda = 1$ and then factorising). Thus, the general solution is $x(t) = c_1e^{-t} + c_2e^t + c_3e^{2t}$.

(iv) The char. polynomial is $\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda + 1)(\lambda - 1)^2 = 0$. Thus, the general solution is $x(t) = c_1e^{-t} + c_2e^t + c_3te^t$.