Solutions 5

Module F13YT2

1. We may write (1) as $L[y] = f(x)$.
Then $u_2 - u_1$ and $u_3 - u_2$ are solutions of $L[y] = 0$, i.e., e^x and 1 are solutions of $L[y] = 0$.
Since $u_1(x)$ is a particular solution of $L[y] = f$, any solution of $L[y] = f$ can be written as $y(x) = x + c_1 e^x + c_2 \times 1$.
Since $y(0) = 1$, we require $1 = 0 + c_1 + c_2$, i.e., $c_1 + c_2 = 1$.
Also $y'(x) = 1 + c_1 e^x$. Hence, as $y'(0) = 3$, we require $3 = 1 + c_1$.
Thus $c_1 = 2$ and $c_2 = -1$ and so the required solution is $y(x) = x + 2e^x - 1$.

2. (i) Let $z = \frac{dy}{dx}$ and consider z as a function of y.
Then $\frac{d^2y}{dx^2} = z\,\frac{dz}{dy}$ and equation becomes $z\,\frac{dz}{dy} = 3y^2$.
Hence $z\,dz = 3y^2\,dy$ and so $\frac{1}{2}z^2 = y^3 + c_1$.
Since $y(0) = 2$ and $y'(0) = 4$, $z = 4$ when $y = 2$ and so $c_1 = 0$.
Hence $\frac{1}{2}z^2 = y^3$, i.e., $(\frac{dy}{dx})^2 = 2y^3$. Thus, as $\frac{dy}{dx}(0) = 4 > 0$, we must solve $\frac{dy}{dx} = +\sqrt{2}y^{\frac{3}{2}}$.
Hence $y^{-\frac{3}{2}}\,dy = \sqrt{2}dx \Rightarrow -2y^{-\frac{1}{2}} = \sqrt{2}x + c_2$.

Since $y(0) = 2$, $c_2 = -\sqrt{2}$.
Hence $-2y^{-\frac{1}{2}} = \sqrt{2}x - \sqrt{2}$ and so $y(x) = \frac{2}{(1-x)^2}$.

(ii) Equation may be written as $z\,\frac{dz}{dx} = x$ where $z = \frac{dy}{dx}$.
Thus $z\,dz = x\,dx$ and so $z^2 = x^2 + c_1$.
Since $z = 1$ when $x = 1$, $c_1 = 0$ and so we must have $(\frac{dy}{dx})^2 = x^2$. Since $\frac{dy}{dx}(1) = 1 > 0$, we must solve $\frac{dy}{dx} = x$. Hence $y = \frac{1}{2}x^2 + c_2$. Since $y(1) = 2$, $c_2 = \frac{3}{2}$ and so $y(x) = \frac{1}{2}x^2 + \frac{3}{2}$.

3. This equation is of Euler form. After the substitution $u = \ln(x)$, we get

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} - 3y = 0.$$

The characteristic equation $\lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1)$ has roots $-3, 1$. Thus the general solution is

$$y(x) = Ax^{-3} + Bx.$$

4. (a) We seek a solution of the form $y(x) = ax^2 + bx + c$.
Then $y'(x) = 2ax + b$ and $y''(x) = 2a$.
Thus we require $2a + 4(2ax + b) + 4(ax^2 + bx + c) = x^2$,
i.e., $4a = 1$, $8a + 4b = 0$, $2a + 4b + 4c = 0$, i.e., $a = \frac{1}{4}$, $b = -\frac{1}{2}$, $c = \frac{3}{8}$.
Thus $y(x) = \frac{1}{8}(2x^2 - 4x + 3)$ is a particular solution.

(b) Since $k = 1$ is a solution of the characteristic equation, we seek a solution of the form $y(x) = kxe^x$.
Then $y'(x) = ke^x + kxe^x$ and $y''(x) = 2ke^x + kxe^x$.

Thus we require $[2k + kx - 4(k + kx) + 3kx]e^x = 2e^x$, i.e., $-2k = 2$, i.e., $k = -1$.
Thus $y(x) = -xe^x$ is a particular solution.

(c) Equation can be written as $y'' + 2y' + 5y = \mathrm{Re}(e^{2ix})$.
We seek a particular solution of the form $y(x) = ce^{2ix}$.
Then $y'(x) = 2ice^{2ix}$ and $y''(x) = -4ce^{2ix}$.
Thus we require $(-4c + 4ic + 5c)e^{2ix} = e^{2ix}$, i.e., $(1 + 4i)c = 1$ and so $c = \frac{1}{1+4i} = \frac{1}{17}(1 - 4i)$.
Thus complex equation has part. soln. $y(x) = \frac{1}{17}(1 - 4i)(\cos(2x) + i\sin(2x))$.
Taking real part we obtain the part. soln. $y(x) = \frac{1}{17}(\cos(2x) + 4\sin(2x))$.

(d) $y''+y = 0$ has fundamental set of solutions $y^{(1)}(x) = \sin(x)$, $y^{(2)}(x) = \cos(x)$. The Wronskian $W(y^{(1)}, y^{(2)}) = -1$. The method of variation of parameters tells us that a particular solution is given by

$$y(x) = c_1(x)\sin(x) + c_2(x)\cos(x).$$

where

$$\begin{aligned} c_1(x) &= \int \cos(x)\sec(x)dx = x \\ c_2(x) &= -\int \sin(x)\sec(x)dx = -\int \tan(x)dx = \ln(\cos(x)). \end{aligned}$$

Thus a particular solution is

$$y(x) = x\sin(x) + \ln(\cos(x))\cos(x).$$

5. (*i*) The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1)$$

with roots $-2, -1$. Thus e^{-2t} is a solution of the homogenous equation and we need to try a particular solution of the form $y_p(x) = cte^{-2t}$. Substituting into the equation, we find

$$ce^{-2t}(4t - 4 + 3(1 - 2t) + 2t) = 5e^{-2t}.$$

Thus $c = -5$, and a particular solution is

$$y_p(t) = -5te^{-2t}$$

. The general solution is therefore

$$y(t) = -5te^{-2t} + Ae^{-2t} + Be^{-t}.$$

(*ii*) The method of variation of parameter tells us that

$$y_p(t) = c_1(t)e^{-2t} + c_2(t)e^{-t}.$$

The Wronksian is given by

$$W(e^{-2t}, e^{-t}) = e^{-3t},$$

and hence the coefficients are

$$\begin{aligned} c_1(t) &= -\int e^{-t}e^{3t}5e^{-2t}dt = -\int 5dt = -5t \\ c_2(t) &= \int e^{-2t}e^{3t}5e^{-2t}dt = \int 5e^{-t}dt = -5e^{-t} \end{aligned}$$

Thus a particular solution is

$$y_p(t) = -5te^{-2t} - 5e^{-2t},$$

and the general solution is

$$y(t) = -5te^{-2t} + Ae^{-2t} + Be^{-t}.$$

6. By inspection $y(x) = x$ is a solution. We seek another solution of the form

$$y(x) = xc(x).$$

Then $y' = c + xc'$ and $y'' = 2c' + xc''$. Thus substituting into the equation we obtain

$$(1 + x^2)(2c' + xc'') - 2x(c + xc') + 2xc = x(1 + x^2)c'' + 2c' = 0.$$

Hence, letting $u = c'$, we get

$$x(1 + x^2)u' + 2u = 0,$$

i.e.,

$$u' + \frac{2}{x(1 + x^2)}u = 0.$$

This is a separable equation; separating variables we find

$$\int \frac{du}{u} = -\int \frac{2}{x(1 + x^2)}\, dx = -\int (\frac{2}{x} - \frac{2x}{1 + x^2})\, dx = -2\ln(x) + \ln(1 + x^2)$$

i.e.,

$$u(x) = \frac{A(1 + x^2)}{x^2}$$

for some constant A. Thus we may choose

$$u(x) = \frac{1}{x^2} + 1$$

and so $c(x)$ is determined by

$$c'(x) = 1 + \frac{1}{x^2},$$

which is solved by

$$c(x) = x - \frac{1}{x}.$$

Thus we obtain another solution

$$y(x) = xc(x) = x^2 - 1$$

and so a fundamental set of solutions is

$$\{x, x^2 - 1\}.$$