Solutions 6

Module F13YT2

1. (i) $\mathcal{L}[(t+1)^2] = \mathcal{L}[t^2+2t+1] = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$.

(ii) $\mathcal{L}[\sin^2(t)] = \mathcal{L}[\frac{1}{2}(1-\cos(2t)] = \frac{1}{2}(\frac{1}{s} - \frac{s}{s^2+4})$.

(iii) $\mathcal{L}[t\sin(t)] = \int_0^\infty t\sin(t)e^{-st}\,dt = \mathrm{Im}(\int_0^\infty te^{it}e^{-st}\,dt)$
$= \mathrm{Im}(\int_0^\infty te^{-(s-i)t}\,dt) = \mathrm{Im}[-\frac{1}{s-i}\,te^{-(s-i)t}|_{t=0}^{t=\infty} + \frac{1}{s-i}\int_0^\infty e^{-(s-i)t}\,dt$
$= \mathrm{Im}[\frac{1}{(s-i)^2}] = \mathrm{Im}[\frac{(s+i)^2}{(s^2+1)^2}] = \frac{2s}{(s^2+1)^2}$.

(iv) Since $\mathcal{L}[\sin(t)] = \frac{1}{s^2+1}$, $\mathcal{L}[e^t\sin(t)] = \frac{1}{(s-1)^2+1} = \frac{1}{s^2-2s+2}$.

(v) Since $\mathcal{L}[t] = \frac{1}{s^2}$, $\mathcal{L}[te^{-t}] = \frac{1}{(s+1)^2}$.

2. (i) Since $\frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1}$ and $\mathcal{L}[\cos(t)] = \frac{s}{s^2+1}$, it follows that $\mathcal{L}^{-1}\{\frac{s+1}{s^2+2s+2}] = e^{-t}\cos(t)$.

(ii) $\frac{1}{s^2-3s+2} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}$ and so $\mathcal{L}^{-1}\{\frac{1}{s^2-3s+2}] = e^{2t} - e^t$.

(iii) If $\frac{s}{(s-1)^2(s^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4}$, then
$s = A(s-1)(s^2+4) + B(s^2+4) + (Cs+D)(s-1)^2$.
$s = 1 \Longrightarrow B = \frac{1}{5}$.
Equating coefficients of s^3 gives $A + C = 0$.
Equating coefficients of s^2 gives $-A + B - 2C + D = 0$ and so $A + 2C - D = \frac{1}{5}$.
Equating coefficients of s^0 gives $-4A + 4B + D = 0$ and so $4A - D = \frac{4}{5}$.
Thus $A = \frac{3}{25}$, $C = -\frac{3}{25}$ and $D = -\frac{8}{25}$.
Hence $\frac{s}{(s-1)^2(s^2+4)} = \frac{3}{25}\frac{1}{s-1} + \frac{1}{5}\frac{1}{(s-1)^2} - \frac{1}{25}\frac{3s+8}{s^2+4}$
$= \frac{3}{25}\frac{1}{s-1} + \frac{1}{5}\frac{1}{(s-1)^2} - \frac{3}{25}\frac{s}{s^2+4} - \frac{4}{25}\frac{2}{s^2+4}$.
Thus $\mathcal{L}^{-1}\{\frac{s}{(s-1)^2(s^2+4)}] = \frac{3}{25}e^t + \frac{1}{5}te^t - \frac{3}{25}\cos(2t) - \frac{4}{25}\sin(2t)$.

(iv) $\frac{s}{s^2+4s+8} = \frac{s}{(s+2)^2+4} = \frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4}$.
Hence $\mathcal{L}^{-1}\{\frac{s}{s^2+4s+8}] = e^{-2t}\cos(2t) - e^{-2t}\sin(2t)$.

3. $y'' - 2y' + 2y = \cos(t); \quad y(0) = 1, \ \ y'(0) = 0.$

Taking Laplace transforms we obtain
$s^2\bar{y}(s) - sy(0) - y'(0) - 2[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{s}{s^2+1}$,
i.e., $(s^2 - 2s + 2)\bar{y}(s) = \frac{s}{s^2+1} + s - 2 = \frac{s^3-2s^2+2s-2}{s^2+1}$.
Thus $\bar{y}(s) = \frac{s^3-2s^2+2s-2}{(s^2+1)(s^2-2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$ where
$(As+B)(s^2-2s+2) + (Cs+D)(s^2+1) = s^3 - 2s^2 + 2s - 2$.
Equating coefficients of s^3 : $A + C = 1$.
Equating coefficients of s^2 : $-2A + B + D = -2$.
Equating coefficients of s : $2A - 2B + C = 2$.
Equating coefficients of s^0 : $2B + D = -2$.
The equations above have solution $A = \frac{1}{5}$, $B = -\frac{2}{5}$, $C = \frac{4}{5}$ and $D = -\frac{6}{5}$.
Hence $\bar{y}(s) = \frac{1}{5}\frac{s-2}{s^2+1} + \frac{2}{5}\frac{2s-3}{(s-1)^2+1} = \frac{1}{5}\frac{s-2}{s^2+1} + \frac{2}{5}\frac{2(s-1)-1}{(s-1)^2+1}$.
Thus $y(t) = \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + \frac{4}{5}e^t\cos(t) - \frac{2}{5}e^t\sin(t)$.

4. $f(t) = u_{\frac{\pi}{2}}(t)\sin(t) = u_{\frac{\pi}{2}}(t)\cos(\frac{\pi}{2} - t) = u_{\frac{\pi}{2}}(t)\cos(t - \frac{\pi}{2})$.

Hence $\overline{f}(s) = e^{-\frac{\pi s}{2}} \frac{s}{s^2+1}$.

$g(t) = [1 - u_2(t)] + 2u_2(t) = 1 + u_2(t)$. Hence $\overline{g}(s) = \frac{1}{s} + \frac{e^{-2s}}{s}$.

5. Equation can be written as

$$y'' + y = 1 - u_1(t) \quad y(0) = 0, \;\; y'(0) = 0.$$

Taking Laplace transforms we obtain
$s^2\bar{y}(s) - sy(0) - y'(0) + \bar{y}(s) = \frac{1}{s} - \frac{e^{-s}}{s}$ and so $\bar{y}(s) = \frac{1}{s(s^2+1)} - e^{-s}\frac{1}{s(s^2+1)}$.
Now $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$ where $A(s^2+1) + (Bs+C)s = 1$,
i.e., $A = 1$, $B = -1$, $C = 0$.
Hence $\bar{y}(s) = (\frac{1}{s} - \frac{s}{s^2+1}) - (\frac{1}{s} - \frac{s}{s^2+1})e^{-s}$.
Thus $y(t) = 1 - \cos(t) - [1 - \cos(t-1)]u_1(t)$.

6. (i) Taking the Laplace transform we obtain

$$(s^2 + s - 6)\bar{y}(s) = 2e^{-s},$$

such that

$$\bar{y}(s) = \frac{2e^{-s}}{(s-2)(s+3)} = \frac{2}{5}e^{-s}\left(\frac{1}{s-2} - \frac{1}{s+3}\right) = \frac{2}{5}e^{-s}\mathcal{L}[e^{2t}] - \frac{2}{5}e^{-s}\mathcal{L}[e^{-3t}].$$

Hence,

$$\begin{aligned} y(t) &= \frac{2}{5}u_1(t)e^{2(t-1)} - \frac{2}{5}u_1(t)e^{-3(t-1)} \\ &= \begin{cases} 0 & t < 1 \\ \frac{2}{5}(e^{2(t-1)} - e^{-3(t-1)}) & t \geq 1 \end{cases}. \end{aligned}$$

(ii) Taking the Laplace transform gives

$$(s^2 + 4s + 4)\bar{y}(s) - sy(0) - y'(0) - 4y(0) = \frac{e^{-3s}}{s}.$$

Thus

$$\bar{y}(s) = \frac{e^{-3s}}{s(s+2)^2} + \frac{2s}{(s+2)^2}.$$

Partial fractions give

$$\begin{aligned} \frac{s}{(s+2)^2} &= \frac{(s+2)-2}{(s+2)^2} = \frac{1}{s+2} - \frac{2}{(s+2)^2} \\ \frac{1}{s(s+2)^2} &= \frac{1}{4}\left(\frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}\right). \end{aligned}$$

Thus

$$\bar{y}(s) = 2\mathcal{L}[e^{-2t}] - 4\mathcal{L}[te^{-2t}] + \frac{1}{4}e^{-3s}(\mathcal{L}[1] - \mathcal{L}[e^{-2t}] - 2\mathcal{L}[te^{-2t}]).$$

Hence

$$y(t) = 2e^{-2t} - 4te^{-2t} + \frac{1}{4}u_3(t)(1 - e^{-2(t-3)} - 2(t-3)e^{-2(t-3)}).$$

7. (i) $\dot{x}_1 = x_1 - x_2, \quad \dot{x}_2 = 5x_1 - 3x_2\ ; \qquad x_1(0) = 1, \quad x_2(0) = 2.$
Taking Laplace transforms we obtain
$s\overline{x}_1(s) - x_1(0) = \overline{x}_1(s) - \overline{x}_2(s);$ i.e., $(s-1)\,\overline{x}_1(s) + \overline{x}_2(s) = 1$ (1)
$s\overline{x}_2(s) - x_2(0) = 5\overline{x}_1(s) - 3\overline{x}_2(s);$ i.e., $-5\overline{x}_1(s) + (s+3)\,\overline{x}_2(s) = 2$ (2)
$5\times(1) + (s-1)\times(2)$ gives $[5 + (s-1)(s+3)]\,\overline{x}_2(s) = 5 + 2(s-1),$
i.e., $(s^2 + 2s + 2)\,\overline{x}_2(s) = 2s + 3,$ i.e., $\overline{x}_2(s) = \frac{2(s+1)}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}.$
Hence $x_2(t) = 2e^{-t}\cos(t) + e^{-t}\sin(t).$
$(s+3)\times(1) - (2)$ gives $[(s+3)(s-1) + 5]\,\overline{x}_1(s) = s + 1,$
i.e., $(s^2 + 2s + 2)\,\overline{x}_1(s) = s + 1,$ i.e., $\overline{x}_1(s) = \frac{s+1}{(s+1)^2+1}.$
Hence $x_1(t) = e^{-t}\cos(t).$

(ii) $\dot{x}_1 = x_1 + x_2 + e^{-2t}; \quad \dot{x}_2 = 4x_1 - 2x_2 - 2e^t\ ; \qquad x_1(0) = 0, \quad x_2(0) = 1.$
Taking Laplace transforms we obtain
$s\overline{x}_1(s) = \overline{x}_1(s) + \overline{x}_2(s) + \frac{1}{s+2};$ i.e., $(s-1)\overline{x}_1(s) - \overline{x}_2(s) = \frac{1}{s+2}$ (1)
$s\overline{x}_2(s) - x_2(0) = 4\overline{x}_1(s) - 2\overline{x}_2(s) - \frac{2}{s-1},$ i.e., $-4\,\overline{x}_1(s) + (s+2)\,\overline{x}_2(s) = 1 - \frac{2}{s-1}$ (2)
$4\times(1) + (s-1)\times(2)$ gives $[-4 + (s-1)(s+2)]\,\overline{x}_2(s) = \frac{4}{s+2} + s - 1 - 2,$
i.e., $(s^2 + s - 6)\,\overline{x}_2(s) = \frac{4}{s+2} + (s-3),$
i.e., $\overline{x}_2(s) = \frac{4+(s-3)(s+2)}{(s+3)(s+2)(s-2)} = \frac{s^2-s-2}{(s+3)(s+2)(s-2)} = \frac{(s+1)(s-2)}{(s+3)(s+2)(s-2)} = \frac{s+1}{(s+3)(s+2)} = \frac{2}{s+3} - \frac{1}{s+2}.$
Hence $x_2(t) = 2e^{-3t} - e^{-2t}.$
$(s+2)\times(1) + (2)$ gives $[(s-1)(s+2) - 4]\,\overline{x}_1(s) = 1 + 1 - \frac{2}{s-1},$
i.e., $(s^2 + s - 6)\overline{x}_1(s) = \frac{2s-4}{s-1},$
i.e., $\overline{x}_1(s) = \frac{2s-4}{(s+3)(s-2)(s-1)} = \frac{2}{(s+3)(s-1)} = -\frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{1}{s-1}.$
Hence $x_1(t) = -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{t}.$

(iii) $\dot{x}_1 = x_2 + f(t); \quad \dot{x}_2 = -x_1 + f(t)\ ; \qquad x_1(0) = 0, \ x_2(0) = 0.$
Taking Laplace transforms we obtain $s\overline{x}_1(s) = \overline{x}_2(s) + \overline{f}(s);$ i.e., $s\overline{x}_1(s) - \overline{x}_2(s) = \overline{f}(s)$
(1)
$s\overline{x}_2(s) = -\overline{x}_1(s) + \overline{f}(s);$ i.e., $\overline{x}_1(s) + s\overline{x}_2(s) = \overline{f}(s)$ (2)
$s\times(2) - (1)$ gives $(s^2+1)\,\overline{x}_2(s) = (s-1)\,\overline{f}(s),$
i.e., $\overline{x}_2(s) = (\frac{s}{s^2+1} - \frac{1}{s^2+1})\,\overline{f}(s) = \overline{g}(s)\,\overline{f}(s)$ where $g(t) = \cos(t) - \sin(t).$
Hence $x_2(t) = \int_0^t f(t-s)\,[\cos(s) - \sin(s)]\,ds.$

$s\times(1) + (2)$ gives $(s^2+1)\,\overline{x}_1(s) = (s+1)\,\overline{f}(s),$
i.e., $\overline{x}_1(s) = (\frac{s}{s^2+1} + \frac{1}{s^2+1})\overline{f}(s) = \overline{g}(s)\overline{f}(s)$ where $g(t) = \cos(t) + \sin(t).$
Hence $x_1(t) = \int_0^t f(t-s)\,[\cos(s) + \sin(s)]\,ds.$