Solutions 7

Module F13YT2

1. (i) The Laplace transform satisfies

$$\bar{y}(s)(s^2+4) = 2\frac{3!}{s^4},$$

and hence

$$\bar{y}(s) = \frac{3!}{s^4}\frac{2}{s^2+4} = \mathcal{L}[t^3]\mathcal{L}[\sin(2t)].$$

Thus

$$y(t) = \int_0^t \tau^3 \sin(2(t-\tau))d\tau = \frac{3}{8}\sin(2t) + \frac{1}{2}t^3 - \frac{3}{4}t.$$

(ii) We proceed as above, and use partial fractions to express

$$\bar{y}(s) = \frac{3!}{s^4}\frac{2}{s^2+4} = \frac{3}{4(s^2+4)} + \frac{3}{s^4} - \frac{3}{4s^2}.$$

Hence,

$$y(t) = \frac{3}{8}\sin(2t) + \frac{1}{2}t^3 - \frac{3}{4}t.$$

(iii) A FSS of the homogeneous equation, $y'' + 4y = 0$ is given by $y^{(1)}(t) = \sin(2t)$, $y^{(2)}(t) = \cos(2t)$. The particular solution of the inhomogeneous equation is of the form $y_p(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$. Substituting into the eqn gives

$$2b_2 + 6b_3 t + 4(b_0 + b_1 t + b_2 t^2 + b_3 t^3) = 2t^3.$$

Hence,

$$b_2 + 2b_0 = 0, \quad 6b_3 + 4b_1 = 0, \quad b_2 = 0, \quad 2b_3 = 1,$$

with solution $b_0 = 0, b_1 = -\frac{3}{4}, b_2 = 0, b_3 = \frac{1}{2}$. The general solution is therefore

$$y(t) = -\frac{3}{4}t + \frac{1}{2}t^3 + A\sin(2t) + B\cos(2t).$$

The initial conditions implies

$$y(0) = 0 \leftrightarrow B = 0, \quad y'(0) = 0 \leftrightarrow -\frac{3}{4} + 2A = 0.$$

Hence, $A = frac38$ and the solution of the initial value problem is

$$y(t) = \frac{3}{8}\sin(2t) + \frac{1}{2}t^3 - \frac{3}{4}t.$$

(iv) The Wronskian associated with the FSS of (iii) is

$$W(y^{(1)}(t), y^{(2)}(t)) = -2$$

Hence, a particular solution is given by $y_p(t) = c_1(t)y^{(1)}(t) + c_2(t)y^{(2)}(t)$, where

$$c_1(t) = \int_0^t \cos(2t)t^3\, dt, \quad c_2(t) = -\int_0^t \sin(2t)t^3\, dt.$$

Performing these, integrals and cancelling numerous terms, we find

$$y_p(t) = \frac{3}{8}\sin(2t) + \frac{1}{2}t^3 - \frac{3}{4}t.$$

One can then proceed to get $y(t)$ as above (note that the fact that $y_p(t) = y(t)$ is coincidental as far as I can see).

2. We have

$$\begin{aligned} F[\delta(x-c)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-c)e^{-ikx}dz = \frac{1}{\sqrt{2\pi}}e^{-ikc}, \\ F[\Delta_\varepsilon(x-c)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Delta_\varepsilon(x-c)e^{-ikx}dz = \frac{1}{\varepsilon\sqrt{2\pi}}\int_{c-\varepsilon/2}^{c+\varepsilon/2} e^{-ikx} = \sqrt{\frac{2}{\pi}}\frac{e^{-ikc}}{\varepsilon k}\sin(k\varepsilon/2). \end{aligned}$$

In the limit, we have

$$\lim_{\varepsilon\to 0} F[\Delta_\varepsilon(x-c)] = \frac{1}{\sqrt{2\pi}}e^{-ikc} = F[\delta(x-c)].$$

3. We can write

$$\begin{aligned} \bar{y}(s) &= \frac{2}{s^2+1} + \frac{s\,e^{-s}}{(s+1)^2+2} - e^{-3s}\frac{1}{s^4} + e^{-2s} \\ &= \frac{2}{s^2+1} + \frac{(s+1)\,e^{-s}}{(s+1)^2+2} - \frac{e^{-s}}{(s+1)^2+2} - e^{-3s}\frac{1}{s^4} + e^{-2s} \\ &= 2\mathcal{L}[\sin(t)] + e^{-s}\mathcal{L}[e^{-t}\cos(\sqrt{2}\,t)] - \frac{e^{-s}}{\sqrt{2}}\mathcal{L}[e^{-t}\sin(\sqrt{2}\,t)] - e^{-3s}\frac{1}{6}\mathcal{L}[t^3] + \mathcal{L}[\delta(t-2)]. \end{aligned}$$

Hence,

$$\begin{aligned} y(t) &= 2\sin(t) + u_1(t)e^{-(t-1)}\cos(\sqrt{2}\,(t-1)) - \frac{1}{\sqrt{2}}u_1(t)e^{-(t-1)}\sin(\sqrt{2}\,(t-1)) \\ &\quad -\frac{1}{6}u_3(t)(t-3)^3 + \delta(t-2). \end{aligned}$$

4. (i) We may write

$$\bar{y}(s) = \frac{s}{(s+3)(s+2)} = \frac{3}{s+3} - \frac{2}{s+2}.$$

Hence,

$$y(t) = 3e^{-3t} - 2e^{-2t}.$$

(ii) We may write

$$\bar{y}(s) = \frac{e^{-3s}2(s+2)}{(s+1)^2+4} = \frac{e^{-3s}2(s+1)}{(s+1)^2+4} + \frac{e^{-3s}2}{(s+1)^2+4} = 2e^{-3s}\mathcal{L}[e^{-t}\cot(2t)] + e^{-3s}\mathcal{L}[e^{-t}\sin(2t)].$$

Hence,

$$y(t) = 2u_3(t)e^{-(t-3)}\cos(2(t-3)) + u_3(t)e^{-(t-3)}\sin(2(t-3))].$$

(iii) We may write

$$\bar{y}(s) = e^{-s}\frac{4(s+1)}{s^2(s+2)^2} = e^{-s}\left(\frac{1}{s^2} - \frac{1}{(s+2)^2}\right) = e^{-s}\mathcal{L}[t] - e^{-s}\mathcal{L}[e^{-2t}t].$$

Hence,

$$y(t) = u_1(t)(t-1) - u_1(t)e^{-2(t-1)}(t-1).$$

5. Taking Laplace transforms then gives

$$\bar{y}(s)(s+3s+2) - (s+3)y(0) - y'(0) = \bar{y}(s)(s+1)(s+2) - 1 = e^{-2s}.$$

Hence,

$$\bar{y}(s) = \frac{1+e^{-2s}}{(s+2)(s+1)}.$$

We may use PF to write

$$\frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2},$$

and so

$$\begin{aligned} \bar{y}(s) &= (1+e^{-2s})\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\ &= (1+e^{-2s})\left(\mathcal{L}[e^{-t}] - \mathcal{L}[e^{-2t}]\right) \end{aligned}$$

Hence,

$$y(t) = (e^{-t} - e^{-2t}) + u_2(t)(e^{-t+2} - e^{-2t+4}).$$

6. Taking the Laplace transform of the system gives

$$\begin{aligned} s\bar{x}_1(s) - x_1(0) &= \bar{x}_1(s) + 2\bar{x}_2(s), \\ s\bar{x}_2(s) - x_2(0) &= 2\bar{x}_1(s) - 2\bar{x}_2(s). \end{aligned}$$

Hence, inserting the initial values, we have

$$\begin{aligned} \bar{x}_1(s)(s-1) - 2\bar{x}_2(s) &= 2, \quad (1) \\ -2\bar{x}_1(s) + \bar{x}_2(s))(s+2) &= 1, \quad (2). \end{aligned}$$

Solving, we find

$$\bar{x}_1(s) = \frac{2}{(s-2)}, \quad \bar{x}_2(s) = \frac{1}{(s-2)}.$$

Hence, $x_1(t) = 2e^{2t}$, $x_2(t) = e^{2t}$.