Solutions 8

Module F13YT2

1. $y'' + 4y = f(x); \quad y(0) = 0, \ \ y'(1) = 0.$
$y'' + 4y = 0$ has solutions $\sin(2x)$ and $\cos(2x)$ and hence we choose
$y_1(x) = \sin(2x)$, $y_2(x) = \cos[2(x-1)] = \cos(2x)\ \cos(2) - \sin(2x)\ \sin(2)$.
Then $W(y_1, y_2)(x) = \det \begin{vmatrix} \sin(2x) & \cos(2x-2) \\ 2\cos(2x) & -2\sin(2x-2) \end{vmatrix}$
$= -2[\sin(2x)\sin(2x-2) - \cos(2x)\cos(2x-2)] = -2\cos(2)$.
Thus Green's function is $G(x,s) = \begin{cases} -\frac{\sin(2s)\ \cos(2x-2)}{2\cos(2)} & \text{if } 0 \le s \le x \\ -\frac{\sin(2x)\ \cos(2s-2)}{2\cos(2)} & \text{if } x \le s \le 1 \end{cases}$
and so solution is
$y(x) = -\frac{1}{2\cos(2)}[\{\int_0^x \sin(2s) f(s)\, ds\} \cos(2x-2) + \{\int_x^1 \cos(2s-2)\, f(s)\, ds\} \sin(2x)]$.

2. (a) $y'' = f(x); \quad y(-1) = 0, \ \ y(1) = 0.$
$y'' = 0$ has solutions 1 and x and hence we choose
$y_1(x) = x + 1$ and $y_2(x) = x - 1$.
Then $W(y_1, y_2)(x) = \det \begin{vmatrix} x+1 & x-1 \\ 1 & 1 \end{vmatrix} = 2$.
Thus Green's function is $G(x,s) = \begin{cases} \frac{(s+1)(x-1)}{2} & \text{if } -1 \le s \le x \\ \frac{(x+1)(s-1)}{2} & \text{if } x \le s \le 1 \end{cases}$

2 (b) From (a) we find

$$\frac{\partial G}{\partial s}(x,s) = \begin{cases} \frac{x-1}{2} & \text{if } -1 \le s \le x \\ \frac{x+1}{2} & \text{if } x \le s \le 1 \end{cases}$$

so that $\frac{\partial G}{\partial s} = \frac{x-1}{2} + u_x(s)$ Hence, by the definition of the Dirac delta function

$$\frac{\partial^2 G}{\partial s^2}(x,s) = \delta(s-x).$$

3. $iii)\ x^2y'' + 4xy' + 2y = x; \quad y(1) + y'(1) = 0, \ \ y(2) + y'(2) = 0.$
$x^2y'' + 4xy' + 2y = x$ is an Euler type equation and so we set $x = e^t$,
i.e., $t = \ln(x)$.
Then $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$.
Hence equation may be rewritten $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$.
Characteristic equation is $k^2 + 3k + 2 = 0$ and so $k = -1$ or $k = -2$.
Thus fundamental set of solutions is $\{e^{-t}, e^{-2t}\}$, i.e., $\{\frac{1}{x}, \frac{1}{x^2}\}$.
Since $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$, we choose $y_1(x) = \frac{1}{x}$.
Since $\frac{d}{dx}(\frac{1}{x^2}) = -\frac{2}{x^3}$, we choose $y_2(x) = \frac{1}{x^2}$.
Then $W(y_1, y_2)(x) = \det \begin{vmatrix} \frac{1}{x} & \frac{1}{x^2} \\ -\frac{1}{x^2} & -\frac{2}{x^3} \end{vmatrix} = -\frac{1}{x^4} = -x^{-4}$.
Thus Green's function is $G(x,s) = \begin{cases} -\frac{s^{-1}x^{-2}}{s^{-4}} & \text{if } 0 \le s \le x \\ -\frac{x^{-1}s^{-2}}{s^{-4}} & \text{if } x \le s \le 1 \end{cases}$,

$$\text{i.e., } G(x,s) = \begin{cases} -\frac{s^3}{x^2} & \text{if } 0 \le s \le x \\ -\frac{s^2}{x} & \text{if } x \le s \le 1 \end{cases}.$$

To apply the general theory we must write the equation as

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = \frac{1}{x}, \quad y(1) + y'(1) = 0 = y(2) + y'(2).$$

Hence solution is

$y(x) = \int_1^2 G(x,s)\frac{1}{s}\,ds = -\int_1^x \frac{s^3}{s}\,ds\,\frac{1}{x^2} - \int_x^2 \frac{s^2}{s}\,ds\frac{1}{x} = -[\int_1^x s^2\,ds]\frac{1}{x^2} - [\int_x^2 s\,ds]\,\frac{1}{x}$
$= \frac{1}{3}(1-x^3)\,\frac{1}{x^2} + \frac{1}{2}(x^2-4)\,\frac{1}{x} = \frac{1}{3x^2} - \frac{2}{x} + \frac{x}{6}.$