Solutions 9

Module F13YT2/YF3 **2006**

1. (*i*) $-y'' = \lambda y; \quad y'(0) = 0, \;\; y'(1) = 0.$
Suppose $\lambda < 0$; then we may write $\lambda = -k^2$ where $k > 0$.
Then $y'' = k^2 y$ has general solution $y = A\cosh(kx) + B\sinh(kx)$.
We have $y' = Ak\sinh(kx) + Bk\cosh(kx)$.
$y'(0) = 0 \iff Bk = 0 \implies B = 0$. Hence
$y'(1) = 0 \iff Ak\sinh(k) = 0$, i.e., $A = 0$.
Thus we must have $A = B = 0$.
Hence, if $\lambda < 0$, $y \equiv 0$ is the only solution, i.e., λ is not an eigenvalue.

Suppose $\lambda = 0$; then equation becomes $y'' = 0$ which has general solution $y = Ax + B$.
Now $y'(0) = y'(1) = 0 \iff A = 0$.
Thus $y(x) \equiv B$ is a solution for any constant B and so $\lambda = 0$ is an eigenvalue.

Suppose $\lambda > 0$; then we may write $\lambda = k^2$ where $k > 0$.
Then $y'' = -k^2 y$ has general solution $y = A\cos(kx) + B\sin(kx)$.
We have $y' = -kA\sin(kx) + kB\cos(kx)$.
$y'(0) = 0 \iff Bk = 0$, i.e., $B = 0$. Hence
$y'(1) = 0 \iff -kA\sin(k) = 0 \iff A = 0$ or $\sin(k) = 0$
$\iff A = 0$ or $k = n\pi$ where $n > 0$ is an integer.
Thus $y = \cos(n\pi x)$ is a nonzero solution corresponding to $\lambda = n^2\pi^2$.
So eigenvalues are $0, \pi^2, 4\pi^2, \ldots$ with eigenfunctions $1, \cos(\pi x), \cos(2\pi x), ..$

(*ii*) $-y'' = \lambda y; \quad y(0) = 0, \;\; y'(1) = 0.$
Suppose $\lambda < 0$; then we may write $\lambda = -k^2$ where $k > 0$.
Then $y'' = k^2 y$ has general solution $y = A\cosh(kx) + B\sinh(kx)$.
We have $y' = Ak\sinh(kx) + Bk\cosh(kx)$.
$y(0) = 0 \iff A = 0$. Hence
$y'(1) = 0 \iff Bk\cosh(k) = 0$, i.e., $B = 0$.
Thus we must have $A = B = 0$.
Hence, if $\lambda < 0$, $y \equiv 0$ is the only solution, i.e., λ is not an eigenvalue.

Suppose $\lambda = 0$; then equation becomes $y'' = 0$ which has general solution $y = Ax + B$.
Now $y(0) = 0 \iff B = 0; \quad y'(0) = 0 \iff A = 0$.
Thus $y \equiv 0$ is the only solution and so $\lambda = 0$ is not an eigenvalue. Suppose $\lambda > 0$; then we may write $\lambda = k^2$ where $k > 0$.
Then $y'' = -k^2 y$ has general solution $y = A\cos(kx) + B\sin(kx)$.
$y(0) = 0 \iff A = 0$. Hence
$y'(1) = 0 \iff kB\cos(k) = 0 \iff B = 0$ or $\cos(k) = 0$
$\iff B = 0$ or $k = \frac{\pi}{2} + n\pi$ for $n = 0, 1, 2 \ldots$
Therefore $y = \sin[(\frac{\pi}{2} + n\pi)x]$ is a nonzero solution corresponding to $\lambda = (\frac{\pi}{2} + n\pi)^2$.
Hence eigenvalues are $\frac{\pi^2}{4}, \frac{9\pi^2}{4}, \frac{25\pi^2}{4}, \ldots$ corresponding to the eigenfunctions $\sin(\frac{\pi x}{2}), \sin(\frac{3\pi x}{2}), \sin(\frac{5\pi x}{2}), \ldots$.

2. The Fourier cosine series of f is given by $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2})$ where

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{0}^{2} f(x)\,dx = \frac{1}{2}\int_{0}^{1} dx = \frac{1}{2}$$

and

$$\begin{aligned} a_n &= \tfrac{1}{2}\int_{-2}^{2} f(x)\cos(\tfrac{n\pi x}{2})\,dx = \int_0^1 \cos(\tfrac{n\pi x}{2})\,dx = \tfrac{2}{n\pi}\sin(\tfrac{n\pi x}{2})|_0^1 \\ &= \tfrac{2}{n\pi}\sin(\tfrac{n\pi}{2}) = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{2}{n\pi} \text{ if } n = 4k+1 \\ -\frac{2}{n\pi} \text{ if } n = 4k+3. \end{cases} \end{aligned}$$

Hence, the Fourier cosine series is

$$\frac{1}{2} + \frac{2}{\pi}\{\cos(\frac{\pi x}{2}) - \frac{1}{3}\cos(\frac{3\pi x}{2}) + \frac{1}{5}\cos(\frac{5\pi x}{2}) - \ldots\}.$$

The Fourier sine series of f is given by $\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{2})$ where

$$\begin{aligned} b_n &= \tfrac{1}{2}\int_{-2}^{2} f(x)\sin(\tfrac{n\pi x}{2})\,dx = \int_0^1 \sin(\tfrac{n\pi x}{2})\,dx = -\tfrac{2}{n\pi}\cos(\tfrac{n\pi x}{2})|_0^1 \\ &= \tfrac{2}{n\pi}(1-\cos(\tfrac{n\pi}{2})) = \begin{cases} 2/n\pi \text{ if } n \text{ is odd} \\ 0 \text{ if } n = 4k \\ 4/n\pi \text{ if } n = 4k+2. \end{cases} \end{aligned}$$

Hence the Fourier sine series is

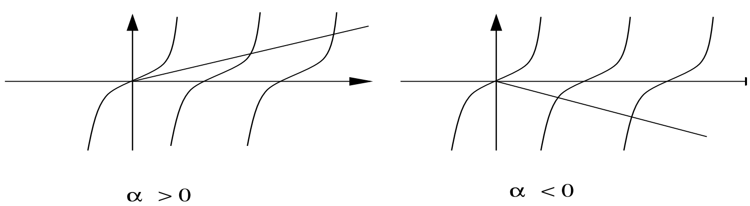
$$\frac{2}{\pi}\{\sin(\frac{\pi x}{2}) + \frac{2}{2}\sin(\frac{2\pi x}{2}) + \frac{1}{3}\sin(\frac{3\pi x}{2}) + \sin(\frac{5\pi x}{2}) + \frac{2}{6}\sin(\frac{6\pi x}{2}) + \ldots\}.$$

3. (i) $-y'' = \lambda y; \quad \alpha y(0) + y'(0) = 0, \;\; y(1) = 0$

(i) First we investigate the existence of positive eigenvalues.
Suppose $\lambda > 0$; $\lambda = k^2$ say.
Then $-y'' = k^2 y$ has general solution $y = A\cos(kx) + B\sin(kx)$.
$\alpha y(0) + y'(0) = 0 \iff \alpha A + kB = 0 \iff B = -\frac{\alpha}{k}A$. Hence
$y(1) = 0 \iff A\cos(k) - \frac{\alpha}{k}A\sin(k) = 0$
$\iff A = 0 \text{ or } k\cos(k) - \alpha\sin(k) = 0.$

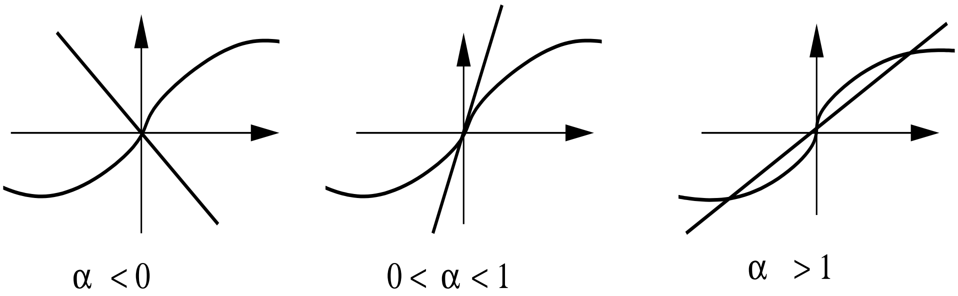
If $\alpha = 0$, we obtain eigenvalues when $\cos(k) = 0$,
i.e., when $k = \frac{\pi}{2} + n\pi$, $n = 0, 1, 2, \ldots$, i.e., $\lambda = \frac{\pi^2}{4}, \frac{9\pi^2}{4}, \frac{25\pi^2}{4}, \ldots$.

If $\alpha \neq 0$, then $k\cos(k) - \alpha\sin(k) = 0 \iff \tan(k) = \frac{k}{\alpha}$.
The line $k \to \frac{k}{\alpha}$ intersects infinitely many of the branches of $k \to \tan(k)$ and so there are infinitely many eigenvalues.



$\alpha > 0$ $\qquad\qquad$ $\alpha < 0$

(ii) and (iii) We now investigate the existence of negative eigenvalues.
Suppose $\lambda < 0$; $\lambda = -k^2$ say.
Then $y'' = k^2 y$ has general solution $y = A\cosh(kx) + B\sinh(kx)$.
Then $\alpha y(0) + y'(0) = 0 \iff \alpha A + kB = 0.$ (1)
Also
$y(1) = 0 \iff A\cosh(k) + B\sinh(k) = 0 \iff A = -\tanh(k)B.$ (2)
Hence we must have $-\alpha\tanh(k)B + kB = 0 \iff B = 0$ or $\frac{k}{\alpha} = \tanh(k)$.
Hence we obtain a negative eigenvalue if and only if the straight line $k \to \frac{k}{\alpha}$ intersects the graph



of $k \to \tanh(k)$ at points $k > 0$. The graphs below show that this does not occur if $\alpha < 0$ or if $0 < \alpha < 1$ but does occur when $\alpha > 1$.
If $\alpha = 0$, it follows easily from equations (1) and (2) above that $A = B = 0$, i.e., there is no negative eigenvalue.

(iv) Finally we investigate the case where $\lambda = 0$.
Then $y'' = 0$ has general solution $y = Ax + B$. Hence
$\alpha y(0) + y'(0) = 0 \iff \alpha B + A = 0.$ (1).
$y(1) = 0 \iff A + B = 0.$ (2)
(1) and (2) have a nonzero solution if and only if $\alpha = 1$.
Thus $\lambda = 0$ is an eigenvalue if and only if $\alpha = 1$.
If $\alpha = 1$, then (1) and (2) have a solution $A = 1$ and $B = -1$ and so we have the eigenfunction $x - 1$.

4. (a) $\int_0^1 1\,dx = 1,\quad \int_0^1 \cos^2(n\pi x)\,dx = \frac{1}{2}\int_0^1 [1 + \cos(2n\pi x)]\,dx = \frac{1}{2}.$
Hence eigenfunction expansion is $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ where
$a_0 = \int_0^1 1\,.\,x\,dx = \frac{1}{2}$
and
$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{n\pi} x\sin(n\pi x)|_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx$
$= \frac{2}{n\pi}\frac{1}{n\pi}\cos(n\pi x)|_0^1 = \begin{cases} 0 \text{ if } n \text{ is even} \\ -\frac{4}{n^2\pi^2} \text{ if } n \text{ is odd.} \end{cases}$
Hence eigenfunction expansion is
$f(x) \sim \frac{1}{2} - \frac{4}{\pi^2}\{\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \ldots\}.$

(b) $\int_0^1 \sin^2(\frac{n\pi x}{2})\,dx = \frac{1}{2}\int_0^1 [1 - \cos(n\pi x)]\,dx = \frac{1}{2}.$
Hence eigenfunction expansion is $f(x) \sim \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{2})$ where
$c_n = 2\int_0^1 x\sin(\frac{n\pi x}{2})\,dx = -2x\frac{2}{n\pi}\cos(\frac{n\pi x}{2})\,|_0^1 + \frac{4}{n\pi}\int_0^1 \cos(\frac{n\pi x}{2})\,dx$
$= -\frac{4}{n\pi}\cos(\frac{n\pi}{2}) + \frac{4}{n\pi}\frac{2}{n\pi}\sin(\frac{n\pi x}{2})\,|_0^1 = -\frac{4}{n\pi}\cos(\frac{n\pi}{2}) + \frac{8}{n^2\pi^2}\sin(\frac{n\pi}{2})$
$= \begin{cases} -4/n\pi & \text{if } n = 4k \\ 8/n^2\pi^2 & \text{if } n = 4k+1 \\ 4/n\pi & \text{if } n = 4k+2 \\ -8/n^2\pi^2 & \text{if } n = 4k+3. \end{cases}$ Thus eigenfunction expansion is
$\frac{8}{\pi^2}\{\sin(\frac{\pi x}{2}) - \frac{1}{9}\sin(\frac{3\pi x}{2}) + \frac{1}{25}\sin(\frac{5\pi x}{2}) + \ldots\} + \frac{4}{\pi}\{\frac{1}{2}\sin(\pi x) - \frac{1}{4}\sin(2\pi x)$
$+ \frac{1}{6}\sin(3\pi x) - \ldots\}.$

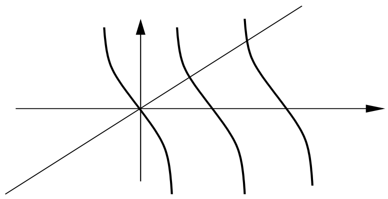
5. $-y'' = \lambda y;\qquad y(0) = 0,\quad y(\pi) + y'(\pi) = 0.$

Suppose $\lambda < 0$, i.e., $\lambda = -k^2$ say.
Then $y'' = k^2 y$ has general solution $y(x) = A\cosh(kx) + B\sinh(kx)$.
$y(0) = 0 \iff A = 0$. Hence
$y(\pi) + y'(\pi) = 0 \iff B\sinh(k\pi) + Bk\cosh(k\pi) = 0$
$\iff B = 0$ or $\sinh(k\pi) + k\cosh(k\pi) = 0 \iff B = 0.$

Hence, if $\lambda < 0$, $y \equiv 0$ is the only solution, i.e., λ is not an eigenvalue.

Suppose $\lambda = 0$, i.e., equation becomes $y'' = 0$ which has general solution $y = Ax + B$.
$y(0) = 0 \Longleftrightarrow B = 0$. Hence
$y(\pi) + y'(\pi) = 0 \Longleftrightarrow A\pi + A = 0 \Longleftrightarrow A = 0$.
Hence $y \equiv 0$ is the only solution and so $\lambda = 0$ is not an eigenvalue.

Suppose $\lambda > 0$, i.e., $\lambda = k^2$ say.
Then $y'' = k^2 y$ has general solution $y(x) = A\cos(kx) + B\sin(kx)$.
$y(0) = 0 \Longleftrightarrow A = 0$. Hence
$y(\pi) + y'(\pi) = 0 \Longleftrightarrow B\sin(k\pi) + Bk\cos(k\pi) \Longleftrightarrow B = 0$ or $k = -\tan(k\pi)$.
The graph below shows that $k = -\tan(k\pi)$ has infinitely many positive solutions $k_1, k_2, \ldots$ corresponding to eigenvalues $\lambda_1 = k_1^2, \lambda_2 = k_2^2, \ldots$.



The corresponding eigenfunctions are $\sin(\sqrt{\lambda_n}x)$.
We now find the eigenfunction expansion of $f(x) = x$ in terms of these eigenfunctions.
$\int_0^\pi \sin^2(\sqrt{\lambda_n}x)\,dx = \frac{1}{2}\int_0^\pi [1 - \cos(2\sqrt{\lambda_n}x)]\,dx = \frac{\pi}{2} - \frac{1}{4\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n}x)|_0^\pi$
$= \frac{\pi}{2} - \frac{1}{4\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n}\pi) = \frac{\pi}{2} - \frac{1}{2\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}\pi)\cos(\sqrt{\lambda_n}\pi)$
$= \frac{1}{2}(\pi + \cos^2(\sqrt{\lambda_n})$ (as $\sin(\sqrt{\lambda_n}\pi) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\pi) = 0$ and so $\sin(\sqrt{\lambda_n}\pi) = -\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\pi)$).
Hence eigenfunction expansion is $f(x) \sim \sum_{n=1}^\infty a_n \sin(\sqrt{\lambda_n}x)$ where
$a_n = \frac{2}{\pi+\cos^2(\sqrt{\lambda_n}\pi)}\int_0^\pi \sin(\sqrt{\lambda_n}x)\,dx = \frac{2}{\pi+\cos^2(\sqrt{\lambda_n}\pi)}\,.\,-\frac{1}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}x)|_0^\pi$
$= \frac{2}{\sqrt{\lambda_n}(\pi+\cos^2(\sqrt{\lambda_n}\pi))}(1 - \cos(\sqrt{\lambda_n}\pi))$.