F1.3YT2/YF3 Ordinary Differential Equations 1& 2

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Contents

1 An Introduction to Differential Equations

1.1 General remarks

The study of differential equations occupies a central place in both applied and pure mathematics. It is fair to say that the subject is the most important part of mathematics for understanding the physical sciences. Applications also abound in other fields, such as engineering, biology and economics. The questions encountered in studying differential equations have in turn inspired many key ideas and concepts in pure mathematics, in particular in analysis. Examples include the theory of Fourier series and Hilbert spaces.

A differential equation is an equation involving a function and its derivatives. If the function depends on one variable only we call it an ordinary differential equation. If it depends on several variables we call it a partial differential equation. In both cases, the order of the differential equation is the order of the highest derivative occurring in it.

Example 1.1.

$\frac{dy}{dx} - xy = 0$ *is a first order ordinary differential equation.*

$\frac{d^2y}{dx^2} - (\frac{dy}{dx})^3 = 3$ *is a second order ordinary differential equation.*

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *is a second order partial differential equation.*

In this course we are only concerned with ordinary differential equations, which I will call ODEs for brevity.

1.2 1st order ODEs

The most general first order ODE is of the form

$$F(x, y, \frac{dy}{dx}) = 0. \tag{1.1}$$

The function $y(x)$ is a solution of (1.1) if

$$F(x, y(x), \frac{dy}{dx}(x)) \equiv 0. \tag{1.2}$$

We call a first order ODE **explicit** if it is of the form

$$\frac{dy}{dx} = f(x, y). \tag{1.3}$$

Otherwise the ODE is called **implicit**.

Example 1.2.

$$\frac{dy}{dx} = 6y + 3x^2 \quad \textit{is explicit,}$$
$$2y(\frac{dy}{dx})^2 + 7x^2\frac{dy}{dx} + 3y = 4 \quad \textit{is implicit.}$$

Before we try to understand the general case, we consider some examples where solutions can be found by elementary methods.

1.2.1 Separable equations

The simplest example of a separable differential equation is of the elementary form

$$\frac{dy}{dx} = f(x). \tag{1.4}$$

The solution is (by definition) by integration

$$y(x) = \int f(x)dx. \tag{1.5}$$

Note that the right hand side is an indefinite integral - determined only up to a constant. As we shall see, solutions of first order differential equations are only determined up to one arbitrary constant.

The solution of an general order ODE containing the maximum number of arbitrary constants is called the **general solution**, and a solution with no arbitrary constants is call a **particular solution**.

Let us compute the general solution of the following slightly more complicated equation:

$$\frac{dy}{dx} = 2xy^2 \tag{1.6}$$

It can be solved by separating the variables, i.e. by bringing all x-dependence to one side and all y-dependence to the other:

$$\int \frac{1}{y^2}dy = \int 2xdx$$

Integrating once yields

$$-\frac{1}{y} = x^2 + c,$$

where c is an arbitrary real constant. Hence, the general solution of (1.6) is

$$y = -\frac{1}{x^2 + c}. \tag{1.7}$$

The constant c is determined if we give the value of y at one point, e.g. at $x = 0$.
Note: In dividing by y we have "lost" one solution: $y \equiv 0$ also solves (1.6), but it is not included in the general family (1.7).

Any equation of the form

$$\frac{dy}{dx} = f(x)g(y) \tag{1.8}$$

can be solved by separating the variables - at least in principle. The solution $y(x)$ is determined implicitly by

$$\int \frac{1}{g(y)} dy = \int f(x) dx. \tag{1.9}$$

In practice it may not be possible to express the integrals in terms of elementary functions and to solve explicitly for y.

1.2.2 Linear equations

A first order explicit equation that can be written in the form

$$\frac{dy}{dx} + a(x)y = b(x), \tag{1.10}$$

is called a **linear** equation.

Consider the following example where, for a change, we have denoted the independent variable by t:

$$\frac{dy}{dt} + 2ty = t. \tag{1.11}$$

First order linear equations can always be solved by using an integrating factor. In the above example, we multiply both sides by $\exp(t^2)$ to obtain

$$e^{t^2} \frac{dy}{dt} + 2te^{t^2} y = te^{t^2}. \tag{1.12}$$

Now the LHS has become a derivative:

$$\frac{d}{dt}(e^{t^2} y) = te^{t^2}. \tag{1.13}$$

This is of the elementary form (1.4). Integrating once yields

$$e^{t^2} y = \frac{1}{2} e^{t^2} + c$$

so that the general solution is

$$y(t) = \frac{1}{2} + ce^{-t^2}. \tag{1.14}$$

More generally, equations of the form

$$\frac{dy}{dt} + a(t)y = b(t), \tag{1.15}$$

where a and b are arbitrary functions of t, can be solved using the integrating factor

$$I(t) = \exp(\int a(t)dt). \tag{1.16}$$

Since the indefinite integral $\int a(t)dt$ is only determined up to an additive constant, the integrating factor is only determined up to a multiplicative constant: if $I(t)$ is an integrating factor, so is $C \cdot I(t)$. Multiplying (1.15) by $I(t)$ we obtain

$$\frac{d}{dt}\left(I(t)y(t)\right) = I(t)b(t). \tag{1.17}$$

Now integrate and solve for $y(t)$ to find the general solution.

1.2.3 Change of variables

Sometimes ODEs can be simplified and solved by changing variables. We illustrate how this works by considering two important classes of ODEs
a)Homogeneous equations

These are equations of the form

$$\frac{dy}{dx} = f(\frac{y}{x}) \tag{1.18}$$

such as a

$$\frac{dy}{dx} = \frac{2xy}{x^2 + y^2} = \frac{2(\frac{y}{x})}{1 + (\frac{y}{x})^2}. \tag{1.19}$$

If we define $u = \frac{y}{x}$ then $y = xu$ and using the product rule we obtain

$$\frac{dy}{dx} = u + x\frac{du}{dx} \tag{1.20}$$

Hence the equation (1.18) can be rewritten as

$$x\frac{du}{dx} = f(u) - u \tag{1.21}$$

which is separable and hence solvable. In our example (1.19) we obtain the following equation for u:

$$x\frac{du}{dx} = \frac{2u}{1 + u^2} - u \tag{1.22}$$

which becomes, after separating variables

$$\int \frac{1 + u^2}{u(1 - u)(1 + u)}du = \int \frac{1}{x}dx \tag{1.23}$$

Using partial fractions

$$\frac{1 + u^2}{u(1 - u)(1 + u)} = \frac{1}{u} + \frac{1}{1 - u} - \frac{1}{1 + u} \tag{1.24}$$

we deduce that u is determined as a function of x by

$$\frac{u}{1-u^2} = cx \tag{1.25}$$

where c is an arbitrary constant. The general solution is thus given implicitly by

$$y = c(x^2 - y^2). \tag{1.26}$$

b) Bernoulli equations (Jakob Bernoulli, 1654-1705)
These are equations of the form

$$\frac{dy}{dx} + a(x)y = b(x)y^\alpha, \tag{1.27}$$

where α is a real number not equal to 1. Let $u = y^{1-\alpha}$. Then

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = (1-\alpha)y^{-\alpha}\frac{dy}{dx}. \tag{1.28}$$

Therefore, (1.27) becomes, after multiplying by $(1-\alpha)y^{-\alpha}$,

$$\frac{du}{dx} + (1-\alpha)a(x)u = (1-\alpha)b(x). \tag{1.29}$$

This is linear and can be solved by finding an integrating factor

Example 1.3. *The equation* $\frac{dy}{dx} + y = y^4$ *becomes a linear equation for* $u = y^{-3}$*:*

$$\frac{du}{dx} - 3u = -3 \tag{1.30}$$

which we can solve using the integrating factor $\exp(-3x)$.

1.2.4 Exact equations

Consider the following equation

$$(\ln x - 2)\frac{dy}{dx} + \frac{y}{x} + 6x = 0. \tag{1.31}$$

If we define

$$\psi(x, y) = y \ln x - 2y + 3x^2 \tag{1.32}$$

then (1.31) can be written

$$\frac{d}{dx}\psi(x, y(x)) = 0 \tag{1.33}$$

and thus solved by

$$\psi(x, y(x)) = c \tag{1.34}$$

for some constant c. Equations which can be written in the form (1.33) for some function ψ are called exact. It is possible to determine whether a general equation of the form

$$a(x,y)\frac{dy}{dx} + b(x,y) = 0 \tag{1.35}$$

is exact as follows. Suppose the above equation were exact. Then we should be able to write it in the form (1.33) for some $\psi : \mathbb{R}^2 \to \mathbb{R}$. However, (1.33) is equivalent to

$$\frac{\partial \psi}{\partial y}\frac{dy}{dx} + \frac{\partial \psi}{\partial x} = 0. \tag{1.36}$$

Thus (1.35) is exact if

$$a(x,y) = \frac{\partial \psi}{\partial y}(x,y) \quad \text{and} \quad b(x,y) = \frac{\partial \psi}{\partial x}(x,y) \tag{1.37}$$

for some function ψ. A necessary condition for the existence of such a function is therefore

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad (\text{since } \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}). \tag{1.38}$$

ψ is then given by integration, i.e., we have both

$$\psi(x,y) = \int a(x,y)\,dy + g(x) \quad \text{and} \quad \psi(x,y) = \int b(x,y)\,dx + h(y)$$

where $g(x)$ and $h(y)$ are arbitrary functions. They are fixed up to an overall constant by requiring compatibility of the two expressions for $\psi(x,y)$.

Example 1.4. $(x + \cos y)\frac{dy}{dx} + y = 0$ *is exact because*

$$\frac{\partial}{\partial x}(x + \cos y) = \frac{\partial y}{\partial y}. \tag{1.39}$$

The function $\psi(x,y)$ is given by requiring both

$$\begin{aligned}
\psi(x,y) &= \int (x + \cos(y))\,dy + g(x) = xy + \sin(y) + g(x) \quad \textit{and} \\
\psi(x,y) &= \int dx + h(y) = xy + h(y).
\end{aligned}$$

Thus we can choose $\psi(x,y) = xy + \sin(y)$ and the general solution of the ODE is $xy + \sin(y) = c$.

Note that it is sometimes possible to make a non-exact equation exact by multiplying with a suitable integrating factor. However, it is only possible to give a recipe for computing the integrating factor in the linear case. In general one has to rely on inspired guesswork.

1.2.5 Existence and uniqueness of solution

Most of the differential equations one encounters in applications cannot be solved by the elementary methods described so far. However, even in the most general case one would still like to know whether a solution exists and whether it is unique. Our experience so far suggest that solutions exist and are unique once we specify the value of the unknown function at one point. We could therefore ask whether the equations

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = \alpha \tag{1.40}$$

has a unique solution. The two equations together are called an **initial value problem**. An important theorem by C.E. Picard (1856-1914) says that, under fairly mild assumption on f, initial value problems have unique solutions, at least locally.

Theorem 1.5. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous in some rectangle $|x - x_0| \leq a, |y - \alpha| \leq b$ and that the partial derivative $\frac{\partial f}{\partial y}$ is also continuous there. Then there is an interval $|x - x_0| \leq h \leq a$ in which the initial value problem (1.40) has a unique solution.*

Remarks

1. Although Picard's theorem guarantees the existence of a solution for a large class of equations, it is not generally possible to find an explicit form of the solution in terms of elementary functions. For example

$$\frac{dy}{dx} = \sin(xy), \qquad y(0) = 1$$

satisfies the condition of Picard's theorem but no explicit formula is known for the solution.

2. If f is not continuous then (1.40) may not have a solution. For example

$$f(x, y) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases} \qquad \text{and} \quad y(0) = 0.$$

This would imply that $y(x) = x$ for $x \geq 0$ and $y(x) = 0$ for $x < 0$, which is not differentiable at $x = 0$.

3. If f does not have a continuous first order partial derivative (1.40) may have more than one solution. For example

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \qquad y(0) = 0$$

is solved by

$$y(x) = \begin{cases} (\frac{2}{3}(x - c))^{\frac{3}{2}} & \text{for} \quad x \geq c \geq 0 \\ 0 & \text{for} \quad x < c \end{cases}$$

for any $c \geq 0$. $y(0) = 0$ gives another solution.

4. Picard's theorem guarantees a solution for x close to x_0, but this solution may not exist for all $x \in \mathbb{R}$:
$$\frac{dy}{dx} = 2xy^2, \qquad y(0) = 1$$
has the solution
$$y(x) = \frac{1}{1-x^2}$$
which tends to infinity as $x \to \pm 1$.

5. Very straightforward looking equations may have no solution:
$$x\frac{dy}{dx} + y = x, \qquad y(0) = 1.$$
Writing the equation as $\frac{d}{dx}(yx) = x$ we deduce $y(x) = x/2 + c/x$, which is incompatible with $y(0) = 1$. Why does Picard's theorem not apply?

1.2.6 Direction fields

For explicit first order equations of the form
$$\frac{dy}{dt} = f(t, y) \tag{1.41}$$
a geometrical viewpoint is helpful for understanding general properties of solutions. Suppose we have a solution curve $y(t)$ and draw its graph in the (t, y)-plane. Then equation (1.41) tells us that the gradient at a point (t, y) on the solution curve is given by $f(t, y)$. To understand the general nature of all solutions it is therefore helpful to draw a short line segment through a number of points (t, y) with slope $f(t, y)$. The collection of all such line segments is called the **direction field** of the differential equation (1.41). Any solution curve must be tangential to the direction field at every point.

Example 1.6. *Sketch the direction field for the differential equation*
$$\frac{dy}{dx} = y(y-1) \tag{1.42}$$
and hence sketch the solutions with initial conditions (i) $y(0) = 0.5$, (ii) $y(0) = 1.1$.

The simplest way to start is to find the values of x and y for which $\frac{dy}{dx} = 0$, $\frac{dy}{dx} > 0$, $\frac{dy}{dx} < 0$. In this question these are respectively $y = 0, 1$, $y > 1$ or $y < 0$, and $0 < y < 1$.

Example 1.7. *Sketch the direction field for the differential equation*
$$\frac{dy}{dx} = x^2 + y^2 \tag{1.43}$$
and hence sketch the solutions with initial conditions (i) $y(0) = 0$.

For more examples of direction fields and methods of sketching them, see the **maple worksheet**.

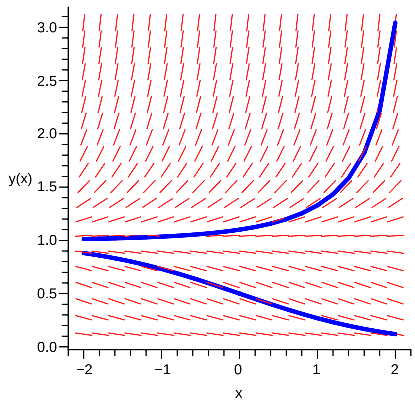

Figure 1.1: Direction field for (1.42)

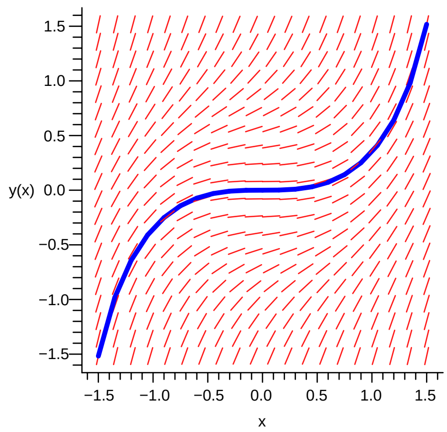

Figure 1.2: Direction field for (1.43)