A Revision of Vector Spaces

Definition

A vector space over a field F (which we shall take to be $\mathbb{R}$) is a set V which is closed under addition and multiplication by a scalar $a \in F$, i.e. if $\mathbf{u}, \boldsymbol{v} \in V$ then $\mathbf{u} + \boldsymbol{v} \in V$ and $a\,\mathbf{u} \in V$. V contains a zero element $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ and $a\,\mathbf{0} = \mathbf{0}$.

Examples

$\mathbb{R}^n$

$C^n(\mathbb{R}, \mathbb{R})$ (the space of real valued functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous and have continuous derivatives $\frac{df}{dx}, \frac{d^2f}{dx^2}, \cdots, \frac{d^nf}{dx^n}$.

Linear Independence

Vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are linearly independent if the only solution of $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots c_n\boldsymbol{v}_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$.

Spanning

Vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n \in V$ span the vector space V if we can write any $\boldsymbol{v} \in V$ as $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ for some choice (not necessary unique) of $c_1, c_2, \cdots, c_n$.
For example: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ span $\mathbb{R}^2$.

Basis

Vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are a basis for V is they are both linearly independent and spanning. In this case, we can write and $\boldsymbol{v} \in V$ as $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ with a unique choice of $c_1, c_2, \cdots, c_n$ (Exercise: prove this).

For a n dimensional vector space, any n linear independent vectors provide a basis. It can be shown that $\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \cdots, \boldsymbol{y}^{(n)} \in \mathbb{R}^n$ are linearly independent if and only if the determinant of the matrix

$$\begin{pmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ & \cdots & & \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{pmatrix}$$

is not equal to zero.

Linear maps

Given two vector spaces V and W over the same field F, one can define linear maps $\phi : V \to W$. These are maps which are compatible with the vector space structure, i.e., they preserve sums and scalar products. We have

$$\phi(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = a\phi(\boldsymbol{v}_1) + b\phi(\boldsymbol{v}_2).$$

A linear map which is a bijection is called a vector space isomorphism.