Integrability in AdS/CFT, the Hubbard Model and Quantum Algebra

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Strong/Weak Interpolation in the AdS/CFT Duality
AdS/CFT Duality

Duality between a string theory on AdS spacetime and CFT on boundary.

Main example:

**IIB strings on** $AdS_5 \times S^5$:
- 2D string sigma model or
- 10D target space model.

**U**($N_c$) $\mathcal{N} = 4$ susy gauge theory:
- 4D conformal QFT,
- technical similarities to QCD.

Two very different models are apparently equivalent.

Parameters:

- $N_c$: string coupling constant $\sim 1/N_c$,
  here: ’t Hooft planar limit $N_c \rightarrow \infty$,
- $\theta$: theta angle, non-perturbative effects,
  here: irrelevant because of large $N_c$,
- $\lambda$: ’t Hooft coupling,
  here: main parameter $\rightarrow$
Spectrum of AdS/CFT

**String Theory**: $AdS_5 \times S^5$ background

States: Solutions $X$ of classical equations of motion plus quantum corrections.

Energy: Charge $E_X$ for translation along AdS-time.

**Gauge Theory**: Conformal $\mathcal{N} = 4$ SYM

States: Local operators. Local, gauge-inv. combinations of the fields, e.g.

$$\mathcal{O} = \text{Tr} \Phi_1 \Phi_2 (\mathcal{D}_1 \mathcal{D}_2 \Phi_2) (\mathcal{D}_1 \mathcal{F}_{24}) + \ldots .$$

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = C |x - y|^{-2D_{\mathcal{O}(\lambda)}} .$$

**AdS/CFT**: String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$?!
**Strong/Weak Duality**

Problem: *Strong/weak duality.*

Perturbative **strings** at $\lambda \to \infty$. Perturbative **gauge theory** at $\lambda \approx 0$.

\[
E(\lambda) = \sqrt{\lambda} E_0 + E_1 + E_2 / \sqrt{\lambda} + \ldots \quad D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \ldots
\]

$E_\ell$: $\ell$ loops (worldsheet model), practical limit: 1 or 2 loops. $D_\ell$: $\ell$ loops (Feynman diagram), practical limit: 3 or 4 loops.

Cannot compare:

- not analytically (term by term),
- not approximately (extrapolation),
- not numerically (lack of method).

Need **finite** $\lambda$ to compare!
Planar Limit

- Simplifications & surprises
- AdS/CFT integrability

**String Theory:** $g_s = 0$.

- Strictly cylindrical worldsheet.
- No string splitting or joining.

**Gauge Theory:** $N_c = \infty$. Only single-trace operators relevant.

- **Translate** single-trace operators to spin chain states, e.g.

\[
O = \text{Tr} \, \phi_1 \phi_1 \phi_2 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_2 \phi_2
\]

\[
|O\rangle = |\uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow \rangle
\]

- **Energy spectrum:** Eigenvalues of spin chain Hamiltonian.
Integrability in AdS/CFT

Planar AdS/CFT models apparently integrable.

**Perturbative String Theory:**
- integrable classical sigma model on supercoset $\frac{PSU(2,2\mid 4)}{Sp(1,1) \times Sp(2)}$, $\text{Arutyunov}$, $\text{Frolov}$, $\text{Staudacher}$
- quantum corrections apparently integrable. $\text{Bena}$, $\text{Polchinski}$, $\text{Roiban}$

**Perturbative Gauge Theory:**
- one loop: integrable NN spin chain Hamiltonian, $\text{Minahan}$, $\text{Zarembo}$, $\text{NB}$, $\text{Staudacher}$ \cite{Minahan2003}
- higher loops: short-range corrections to Hamiltonian. $\text{Kristjansen}$, $\text{Staudacher}$ \cite{Kristjansen2003}

Integrability enables precise computations:
Using constructions and educated guesses based on integrability proposed:
- all-order expansion at $\lambda \to \infty$, $\text{NB, Staudacher}$ \cite{NBStaudacher2005}
- all-order expansion at $\lambda \approx 0$. $\text{NB, Eden}$, $\text{Staudacher}$ \cite{NBStaudacher2005}

Small window of numerical overlap!
Strong/Weak Interpolation

Convergence Properties:
Expansion at $\lambda \to \infty$
- Series asymptotic, no convergence,
- good approximation at low orders.
Expansion at $\lambda \approx 0$
- finite radius of convergence $|\lambda| < \pi^2$,
- defines holomorphic function.

Finite Coupling:
- Integral representations exist,
- numerical evaluation convenient.
Finite coupling model:
- quantum sigma model!
- long-range spin chain?!
- quantum algebra?

perturbative
AdS$_5 \times$S$^5$ strings

classical sigma model
quantised sigma model
quantum sigma model
finite coupling
Bethe Ansatz
Short-Range Spin Chain

Action of perturbative spin chain Hamiltonian at \( g \sim \sqrt{\lambda} \approx 0 \)

\[
\mathcal{H}(g) = g^0 + g^2 + g^3 + g^3 + g^4 + \ldots
\]

- \( \mathcal{O}(g^0) \): excitation number operator
- \( \mathcal{O}(g^2) \): nearest-neighbour Hamiltonian (supersymmetric XXX\(_{1/2}\))
- \( \mathcal{O}(g^4) \): next-nearest-neighbour corrections, range increases with order
- \( \mathcal{O}(g^3) \): length fluctuates . . .

At least: excitation number \( \mathcal{H}_0 \) conserved, local action (on long chains)!

Representation of symmetry generators \( \mathfrak{psu}(2, 2|4) \)

\[
\mathcal{J}(g) = g^0 + g^1 + g^1 + g^2 + \ldots
\]

Not a coproduct! (or not clear how to interpret as such)
Asymptotic Bethe Ansatz

Spectrum? States of infinite spin chain with few “excitations”:

- **Ferromagnetic vacuum**: \( |0\rangle = |\ldots 000\ldots \rangle \), all constituent particles ‘0’.
- **One-magnon states** with excitation \( 0 \rightarrow 1_A \) of momentum \( p \)

\[
|A, p\rangle = \sum_a e^{ipa} |\ldots 0\ldots 1_A\ldots 0\ldots \rangle, \quad \mathcal{H} |A, p\rangle = E(p) |A, p\rangle.
\]

(8|8) admissible flavours \( 1_A \) of single excitations.
- **Asymptotic two-magnon states**, Hamiltonian eigenvalue \( E(p) + E(q) \)

\[
|A, p; B, q\rangle \simeq \sum_a \ll b \quad e^{ipa+iqb} |\ldots 0\ldots 1_A\ldots 1_B\ldots 0\ldots \rangle + \sum_a \gg b \quad \downarrow \\
\quad + S_{AB}^{CD}(p, q) \sum_{a \gg b} e^{ipa+iqb} |\ldots 0\ldots 1_D\ldots 1_C\ldots 0\ldots \rangle.
\]

- **Factorised scattering for three or more magnons** (?!).
Residual Symmetry

QM particle model of 8 bosonic and 8 fermionic flavours on the circle. Integrability: S-matrix as R-matrix of quasi-triangular Hopf algebra?!

Excitations transform as \((2|2) \times (2|2)\) of \(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)\).

Consider just \((2|2)\) flavours and one copy of \(\mathfrak{psu}(2|2)\). Generators:

- \(R_{ab}\): \(\mathfrak{su}(2)\) subalgebra of internal symmetry \(\mathfrak{su}(4)\).
- \(L^{\alpha \beta}\): \(\mathfrak{su}(2)\) subalgebra of conformal symmetry \(\mathfrak{su}(2,2)\).
- \(Q_{ab}\): 4 (Poincaré) supercharges.
- \(S_{\alpha \beta}\): 4 (conformal) supercharges.

\(\mathfrak{psu}(2|2)\) has three-dimensional (exceptional!) central extension.

Need this central extension \(\mathfrak{h} := \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3\) for consistency:

- \(\mathcal{C}\): Hamiltonian (up to integer shift),
- \(\mathcal{P}\): (classical) gauge variation,
- \(\mathcal{K}\): (quantum) gauge variation.
Lie Algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ and Coproduct

Commutators define Lie algebra

- $\mathcal{R}^a_b, \mathcal{L}^\alpha_\beta$: canonical brackets of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ generators,
- $\mathcal{C}, \mathcal{P}, \mathcal{K}$: central elements,
- $\mathcal{Q}^\alpha_b, \mathcal{S}^a_\beta$: supercharges

\[
\{ \mathcal{Q}^\alpha_b, \mathcal{S}^c_\delta \} = \delta^c_b \mathcal{L}^\alpha_\delta + \delta^\alpha_\delta \mathcal{R}^c_b + \delta^c_b \delta^\alpha_\delta \mathcal{C},
\]
\[
\{ \mathcal{Q}^\alpha_b, \mathcal{Q}^\gamma_d \} = \epsilon^{\alpha\gamma} \epsilon_{bd} \mathcal{P},
\]
\[
\{ \mathcal{S}^a_\beta, \mathcal{S}^c_\delta \} = \epsilon^{ac} \epsilon_{\beta\delta} \mathcal{K}.
\]

Length fluctuations lead to non-trivial coproduct

\[
\Delta(\mathcal{J}^A) = \mathcal{J}^A \otimes 1 + \mathcal{U}^{[A]} \otimes \mathcal{J}^A
\]

with $[\mathcal{P}] = +2$, $[\mathcal{Q}] = +1$, $[\mathcal{K}] = [\mathcal{L}] = [\mathcal{C}] = 0$, $[\mathcal{S}] = -1$, $[\mathcal{R}] = -2$. Abelian group-like generator $\mathcal{U}$ measures magnon momentum $e^{ip/2}$.

Cocommutativity on centre: $\mathcal{P} = g\alpha^+ (1 - \mathcal{U}^2)$, $\mathcal{K} = g\alpha^{-1} (1 - \mathcal{U}^{-2})$. 

Fundamental Representation

Have $(2|2)$ flavours of particles $\{|\phi^a\rangle, |\psi^\alpha\rangle\}$. Represent algebra!

Most general action compatible with $su(2) \times su(2)$

\[
\begin{align*}
\mathcal{Q}^\alpha_b |\phi^c\rangle &= a \delta^c_b |\psi^\alpha\rangle, \\
\mathcal{S}^\alpha_b |\psi^\gamma\rangle &= b \varepsilon^{\alpha\gamma} \varepsilon_{bd} |\phi^d\rangle,
\end{align*}
\]

Imposing consistency of superalgebra

- fixes central charges $C = \frac{1}{2}(ad + bc)$, $P = ab$, $K = cd$,
- yields constraint $ad - bc = 1$ or $C^2 - PK = \frac{1}{4}$.

Cocommutativity constraints $P, K = g\alpha^{\pm 1}(1 - \mathcal{U}^{\pm 2})$

- provide dispersion relation $C^2 = \frac{1}{4} + 4g^2 \sin^2(\frac{1}{2}p)$.
- Lattice-like (Brillouin zones) and almost relativistic: Deformed Poincaré.
- Elliptic curve with modulus $k = 4ig$ (rectangular complex torus).
Fundamental R-Matrix

Ansatz for fundamental R-matrix with $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry

\[ R|\phi^a \phi^b\rangle = A_{12}|\phi^{\{a} \phi^{b\}}\rangle - B_{12}|\phi^{[a} \phi^{b]}\rangle + \frac{1}{2} C_{12} \varepsilon^{a b} \varepsilon_{\gamma \delta} |\psi^\gamma \psi^\delta\rangle, \]
\[ R|\psi^\alpha \psi^\beta\rangle = -D_{12}|\psi^{\{\alpha} \psi^{\beta\}}\rangle + E_{12}|\psi^{[\alpha} \psi^{\beta]}\rangle - \frac{1}{2} F_{12} \varepsilon^{\alpha \beta} \varepsilon_{cd} |\phi^c \phi^d\rangle, \]
\[ R|\phi^a \psi^\beta\rangle = G_{12}|\phi^a \psi^\beta\rangle + H_{12}|\psi^\beta \phi^a\rangle, \]
\[ R|\psi^\alpha \phi^b\rangle = K_{12}|\phi^b \psi^\alpha\rangle + L_{12}|\psi^\alpha \phi^b\rangle. \]

Invariance $R \circ \Delta(\hat{J}^A) = \Delta_{op}(\hat{J}^A) \circ R$ fixes $A, \ldots, L$ up to phase function.

YBE $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ fulfilled.

Questions to be addressed:

- What spin chain model does this R-matrix generate?
- What is the Hopf algebra and its universal R-matrix?

Steps: ★ Quantum deformation? ★ Yangian? ★ Classical r-matrix?
Hubbard Model
One-Dimensional Hubbard Model

Electronic model: Fermionic oscillator $c_s, c_s^\dagger$, four states per site

\[ |\circ\rangle = |0\rangle, \quad |\uparrow\rangle = c_{\uparrow}^\dagger |0\rangle, \quad |\downarrow\rangle = c_{\downarrow}^\dagger |0\rangle, \quad |\updownarrow\rangle = c_{\uparrow}^\dagger c_{\downarrow}^\dagger |0\rangle. \]

Simple nearest-neighbour Hamiltonian with $su(2) \times su(2)$ symmetry

\[
H_{\text{Hub}}^{j,k} = \sum_{\alpha=\uparrow,\downarrow} \left( c_{\alpha,j}^\dagger c_{\alpha,k} + c_{\alpha,k}^\dagger c_{\alpha,j} \right) + U c_{\uparrow,j}^\dagger c_{\downarrow,j}^\dagger c_{\downarrow,j} c_{\uparrow,j}. 
\]

Diagonalised by Lieb–Wu equations and $E = \sum_k \cos k_k$

\[
1 = \exp(-ik_k K) \prod_{j=1}^M \frac{2 \sin k_k - 2 \Lambda_j + \frac{i}{2} U}{2 \sin k_k - 2 \Lambda_j - \frac{i}{2} U},
\]

\[
1 = \prod_{j=1}^N \frac{2 \Lambda_k - 2 \sin k_j + \frac{i}{2} U}{2 \Lambda_k - 2 \sin k_j - \frac{i}{2} U} \prod_{j=1}^M \frac{2 \Lambda_k - 2 \Lambda_j - iU}{2 \Lambda_k - 2 \Lambda_j + iU}. 
\]
Shastry’s R-Matrix

One-dimensional Hubbard model has an interesting R-matrix

- Intricate form with around 10 coefficient functions.
- Not of difference form $R(u_1, u_2) \neq R(u_1 - u_2)$.
- Spectral parameters $u_k$ defined on elliptic curve.
- Does not fit standard scheme $Y(g)$ for simple Lie (super)algebras $g$.
- Exceptional integrable structure?!

Consider the identification of bosonic/fermionic states

$$|\circ\rangle = |\phi^1\rangle, \quad |\uparrow\rangle = |\psi^1\rangle, \quad |\downarrow\rangle = |\psi^2\rangle, \quad |\uparrow\downarrow\rangle = |\phi^2\rangle.$$  

Integrable structures agree:

- Bethe equations can be made to coincide (choice of Bethe roots).
- Fundamental R-matrix equivalent to R-matrix of Hubbard chain.
- Hubbard chain integrability explained, simple construction for $R$.
- Exceptional case of centrally extended $\mathfrak{psu}(2|2)$. 
Quantum Deformation
Quantum Deformations $U_q(psu(2|2) \rtimes \mathbb{R}^3)$

Can we quantum deform all of the above?

- More convenient to formulate Hopf algebra & R-matrix at $q \neq 1$.
- Yangian double as contraction of quantum affine algebra at $q \rightarrow 1$.
- Quantum deformation of the Hubbard model?

Use **Chevalley–Serre basis**: $E_k, F_k, H_k$, $k = 1, 2, 3$

Drop two Serre relations for central extension (consistent with coalgebra!)

\[
\{ [E_1, E_2]_q, [E_3, E_2]_q \} = \mathcal{P} \neq 0, \quad \{ [F_1, F_2]_q, [F_3, F_2]_q \} = \mathcal{K} \neq 0.
\]

Construction of algebra and coalgebra standard, but deform by $\Delta$:

\[
\Delta(E_2) = E_2 \otimes 1 + q^{-H_k} \mathcal{U}^+ \otimes E_2, \quad \Delta(F_2) = F_2 \otimes q^H \mathcal{U}^{-1} \otimes F_2.
\]
Fundamental Representation

Can construct a \((2|2)\)-dimensional representation as before:

- Everything quantum-deformed, e.g. constraint \([C]_q^2 - PK = \frac{[1]}{2}_q^2\).
- Still rectangular elliptic curve with \(k = 4ig\sqrt{1 - g^2(q - q^{-1})^2}\).

Invariant fundamental R-matrix can be constructed.

- Big mess (bi-elliptic functions).
- Satisfies YBE!
- \(q = 1\) limit is previous fundamental R-matrix.

Three-parametric \((g, q, u)\) NN integrable spin chain Hamiltonian

\[
\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1}, \quad \mathcal{H}_{12} = -i \left. \mathcal{R}_{12}^{-1} \frac{d}{du_1} \mathcal{R}_{12} \right|_{u_{12}=u}.
\]

Hamiltonian includes Alcaraz–Bariev deformation of Hubbard model.
**Yangian Y(\(\text{psu}(2|2) \ltimes \mathbb{R}^3\))**

Can we find a Yangian generator \(\hat{J}^A\) to enhance \(J^A\) of \(\text{psu}(2|2) \ltimes \mathbb{R}^3\)?

Coproduct of \(\hat{J}^A\) is deformed by \(U\)

\[
\Delta(\hat{J}^A) = \hat{J}^A \otimes 1 + U^{[A]} \otimes \hat{J}^A.
\]

Educated guess for deformed coproduct of \(\hat{\hat{J}}^A\)

\[
\Delta(\hat{\hat{J}}^A) = \hat{\hat{J}}^A \otimes 1 + U^{[A]} \otimes \hat{\hat{J}}^A - \frac{i}{2} f_{BC}^{A} \hat{\hat{J}}^{B} U^{[C]} \otimes \hat{\hat{J}}^{C}.
\]

Define evaluation representation

\[
\hat{\hat{J}}^A |X, p\rangle = \frac{1}{2} \cot(\frac{1}{2}p) \sqrt{1 + 16g^2 \sin^2(\frac{1}{2}p)} \hat{\hat{J}}^A |X, p\rangle.
\]

Fundamental R-matrix is invariant under Yangian!

- Hopf algebra probably Yangian double generated by \(\hat{J}^A_n\), \(n \in \mathbb{Z}\).
- Presumably \(q = 1\) contraction of quantum affine algebra.
- One more symmetry discovered. How to complete algebra?
Classical Limit
Classical r-matrix & Lie Bialgebra

R-matrix has a classical limit $g \to \infty$, $p \simeq g^{-1}$

$$\mathcal{R}_{12} = 1 \otimes 1 + g^{-1}r_{12} + \mathcal{O}(g^{-2})$$

with classical r-matrix $r$. Coproduct becomes cobracket $\delta$

$$\Delta(\hat{J}) - \Delta_{\text{op}}(\hat{J}) = g^{-1}\delta(\hat{J}) + \mathcal{O}(g^{-2}).$$

Treatment simplifies:

- Need only loop algebra $\mathfrak{g}$. No universal enveloping. No deformation.
- Classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$.
- Cobracket given by $\delta(\hat{J}^A_n) = [\hat{J}^A_n, r]$.
- Classical YBE: $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.

Proposal for deformed $u(2|2)$ loop algebra & r-matrix.
Deformation of the $u(2|2)$ Loop Algebra

Alternative proposal for deformed ($\beta$) $u(2|2)$ loop algebra

\[
\{(\mathcal{Q}_m)^\alpha_b, (\mathcal{S}_n)^c_\delta\} = \delta^c_b (\mathcal{L}_{m+n})^\alpha_\delta + \delta^\alpha_\delta (\mathcal{R}_{m+n})^c_b + \delta^c_\delta \delta^\alpha (\mathcal{C}_{m+n}),
\]

\[
\{(\mathcal{Q}_m)^\alpha_b, (\mathcal{Q}_n)^\gamma_d\} = +2\alpha\beta \varepsilon^{\alpha\gamma} \varepsilon_{bd} \mathcal{C}_{m+n-1},
\]

\[
\{(\mathcal{S}_m)^a_\beta, (\mathcal{S}_n)^c_\delta\} = -2\alpha^{-1} \beta \varepsilon^{ac} \varepsilon_{\beta \delta} \mathcal{C}_{m+n-1},
\]

\[
[\mathcal{B}_m, (\mathcal{Q}_n)^\alpha_b] = + (\mathcal{Q}_{m+n})^\alpha_b - 2\alpha\beta \varepsilon^{\alpha\gamma} \varepsilon_{bd} (\mathcal{S}_{m+n-1})^d_\gamma,
\]

\[
[\mathcal{B}_m, (\mathcal{S}_n)^a_\beta] = - (\mathcal{S}_{m+n})^a_\beta - 2\alpha^{-1} \beta \varepsilon^{ac} \varepsilon_{\beta \delta} (\mathcal{Q}_{m+n-1})^\delta_c.
\]

Properties:

- Satisfies Jacobi identities.
- Non-homogeneous loop level.
- Includes additional (automorphism) generators $\mathcal{B}_m$.
- Central extensions $\mathcal{P}, \mathcal{R}$ (and thus $\mathcal{U}$) replaced by loop charge $\mathcal{C}_{-1}$. 
Classical r-matrix

Classical r-matrix is almost standard $u(2|2)$ r-matrix

$$r = r_{\text{psu}(2|2)} - \sum_{m=-1}^{\infty} \mathcal{B}_{-1-m} \otimes \mathcal{C}_m - \sum_{m=+1}^{\infty} \mathcal{C}_{-1-m} \otimes \mathcal{B}_m$$

with the classical r-matrix $r_{\text{psu}(2|2)}$ for $\text{psu}(2|2)$

$$r_{\text{psu}(2|2)} = + \sum_{m=0}^{\infty} (\mathcal{R}_{-1-m})^c_d \otimes (\mathcal{R}_m)^d_c - \sum_{m=0}^{\infty} (\mathcal{L}_{-1-m})^\gamma_\delta \otimes (\mathcal{L}_m)^\delta_\gamma$$

$$+ \sum_{m=0}^{\infty} (\mathcal{Q}_{-1-m})^\gamma_\delta \otimes (\mathcal{Q}_m)^\delta_c - \sum_{m=0}^{\infty} (\mathcal{S}_{-1-m})^c_\delta \otimes (\mathcal{S}_m)^\delta_c.$$

Satisfies CYBE $\Rightarrow$ quasi-triangular Lie bialgebra.

Double structure $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+$ with subalgebra decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$: Standard $\mathfrak{J}_{n \geq 0} \in \mathfrak{g}_+$ (but $\mathcal{B}_0 \in \mathfrak{g}_-$) and $\mathfrak{J}_{n < 0} \in \mathfrak{g}_-$ (but $\mathcal{C}_{-1} \in \mathfrak{g}_+$).
Restriction of Maximally Extended Algebra

Can start with maximally extended algebra $\mathfrak{h}_+ = \mathfrak{sl}(2) \ltimes \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$.

Has r-matrix

$$r_{\mathfrak{h}_+} = r_{\mathfrak{psu}(2|2)} - \sum_{m=0}^{\infty} (\mathcal{B}_{-1-m})^c_d \otimes (\mathcal{C}_m)^d_c - \sum_{m=0}^{\infty} (\mathcal{C}_{-1-m})^c_d \otimes (\mathcal{B}_m)^d_c.$$ 

Restrict automorphisms $(\mathcal{B}_m)^a_b$ to subalgebra spanned by

$$\mathcal{B}_n = (\mathcal{B}_n)^1_1 - (\mathcal{B}_n)^2_2 + 2\alpha^{-1}\beta(\mathcal{B}_{n-1})^1_2 + 2\alpha\beta(\mathcal{B}_{n-1})^2_1.$$ 

Factor out an ideal spanned by some $(\mathcal{C}_m)^a_b$ such that

$$\mathcal{C}_n = (\mathcal{C}_n)^1_1 = -(\mathcal{C}_n)^2_2 = (\mathcal{C}_{n+1})^1_2/2\alpha\beta = (\mathcal{C}_{n+1})^2_1/2\alpha^{-1}\beta.$$ 

Modified r-matrix belongs to $\mathfrak{g} \otimes \mathfrak{g}$ and satisfies CYBE!

$$r := r_{\mathfrak{h}_+} + (\mathcal{C}_{-1})^1_1 \wedge ((\mathcal{B}_0)^1_1 - (\mathcal{B}_0)^2_2) \in \mathfrak{g} \otimes \mathfrak{g}.$$
Conclusions
Conclusions

★ Extended $\mathfrak{psu}(2|2)$ Algebra
- Symmetry for AdS/CFT scattering picture.
- Hopf algebra structure yields interesting fundamental R-matrix.
- Quantum deformation $U_q(\mathfrak{psu}(2|2))$ appears to work.
- Integrable structure of Hubbard model and AB deformation explained.
- R-matrix appears to have Yangian symmetry.
- Classical bialgebra and $r$-matrix identified. Agrees with classical limit of R-matrix.

★ Outlook
- Find Yangian double (quantum affine algebra) and universal R-matrix.
- Understand integrable structure for long-range $\mathfrak{psu}(2,2|4)$ chain.