Dynamics of Ultracold Gases
in one spatial dimension

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Equilibration
It seems 2b good to work in 1D.

Because it's just a bit...

...complicated to keep track of dynamics in 3D.
Elastic scattering of classical point-like particles in $d > 1$ dimensions
Elastic scattering of classical point-like particles in one dimension

Before collision:

\[ p \rightarrow -p \]

After collision:

\[ -p \rightarrow p \]
Elastic scattering of quantum point-like particles in one dimension

Before collision:

\[ p \rightarrow -p \]

After collision:

\[ -p \rightarrow p \]
Long-time dynamics of a 1D quantum gas?

A quantum Newton's cradle.

[T. Kinoshita, T. Wenger, and D. S. Weiss, Nature 440 (06)]

Indication for strong suppression of damping
Thermalisation of a closed quantum system
Thermalisation of a classical system

*N* particles in *d* dimensions: \((2Nd)\)-dimensional phase space

Condition for thermalization:

**Ergodicity**
- phase-space averages = time averages
- phase-space trajectories trace out energy hypersurface uniformly

Ergodicity and Thermalisation in a quantum system?

QM: Dephasing Recurrence
Thermalisation of a quantum system

v. Neumann (1929): “Quantum Ergodicity”

Def.: 
\[ \hat{\rho}_{mc}(E) = \sum_{\alpha \in \mathcal{H}(E)} \frac{1}{N} |\Psi_\alpha \rangle \langle \Psi_\alpha| \]

of \( N \) states on Hilbert subspace \( \mathcal{H}(E) \) on energy shell between \( E \) and \( E + \delta E \)

However:

From initial state \( |\Psi_0 \rangle = \sum_{\alpha \in \mathcal{H}(E)} c_\alpha |\Psi_\alpha \rangle \)

one obtains, at late times, the time average

\[ \overline{|\Psi(t)\rangle \langle \Psi(t)|} = \sum_{\alpha} |c_\alpha|^2 |\Psi_\alpha \rangle \langle \Psi_\alpha| = \hat{\rho}_{\text{diag}} \]

(“diagonal ensemble”)

[Ergodicity requires \( |c_\alpha|^2 = 1/N \)]

[Rigol et al., PRL 98 (07); Polkovnikov et al., RMP 83 (11)]
Thermalisation of a quantum system


Take:
\[
\langle \Psi(t)|M_\beta(t)|\Psi(t)\rangle = \text{Tr}[M_\beta \hat{\rho}_{\text{diag}}] = \langle M_\beta \rangle_{\text{mc}}
\]

for set of macroscopic observables \( \{M_\beta\} \) (coarse-grained on \( \mathcal{H}(E) \) & commuting)

Eigenstate Thermalisation Hypothesis

[Deutsch, PRA 43 (91); Srednicki, PRE 50 (94)]

[Polkovnikov et al., RMP 83 (11)]
Experiment: Ergodicity is not guaranteed!

– if the closed system is (nearly) integrable.

[T. Kinoshita, T. Wenger, and D. S. Weiss, Nature 440 (06)]

Indication for strong suppression of damping
Interacting 1D Bose gas
Lieb-Liniger model: integrable

Lieb-Liniger Bose gas in 1D:

\[ i \frac{\partial \psi_B}{\partial t} = - \sum_{i=1}^{N} \frac{\partial^2 \psi_B}{\partial x_i^2} + \sum_{1 \leq i < j \leq N} 2c \delta(x_i - x_j) \psi_B \]

Solution via Bethe-Ansatz with cusp conditions @ particle contact hyperplanes

\[ x_i = x_j \quad \text{for} \quad c > 0 \]

Dirichlet-von Neumann

Dynamics: H. Buljan, R. Pezer, & TG, PRL 100 (2008)
Lieb-Liniger model: integrable

Lieb-Liniger Bose gas in 1D:

\[ i \frac{\partial \psi_B}{\partial t} = - \sum_{i=1}^{N} \frac{\partial^2 \psi_B}{\partial x_i^2} + \sum_{1 \leq i < j \leq N} 2c \delta(x_i - x_j) \psi_B \]

Quantum integrable:

- Solutions derived by Bethe Ansatz

\textit{N integrals of motion for N particles}
Generalised Gibbs Ensemble (GGE)

Jaynes' maximum-entropy principle \[ \text{[PR 106, 620 (57)]} : \]

implies GGE at \( t \to \infty \):

\[
\hat{\rho} = Z^{-1} \exp \left[ - \sum_m \lambda_m \hat{I}_m \right]
\]

\[
Z = \text{Tr}[\exp(-\sum_m \lambda_m \hat{I}_m)]
\]

\[
\text{Tr}[\hat{I}_m \hat{\rho}] = \langle \hat{I}_m \rangle(t = 0) \quad \text{fixes} \quad \{ \lambda_m \}
\]

Maximisation of entropy under constraints for macroscopic observables.

Rigol et al., PRL 98, 050405 (07), P. Reimann, PRL 101, 190403 (08).
1D Bose gas @ low energies
1D Bose gas

Many-body Hamiltonian:

\[ H = \int dx \left[ -\Phi_x^\dagger \frac{\partial^2}{\partial x^2} \Phi_x + \frac{g}{2} \Phi_x^\dagger \Phi_x^\dagger \Phi_x \Phi_x \right] \]

1D Coupling:

\[ g = \frac{2a}{ml_{\perp}^2} \]

transverse oscillator length
Bogoliubov sound

Linearisation around mean field $\phi_x = \langle \Phi_x \rangle$

\[ \omega_q^2 = c_s^2 q^2 + \left(\frac{q^2}{2m}\right)^2 \]

(sound/particle dispersion: \quad (\text{quasi 1D trap: } \omega_z \ll \omega_\perp) \]

transverse exc.

sound

momentum $q$
Equilibration from dynamical Quantum Field Theory
Dynamical Quantum Field Theory

\[ [\Phi(t, x), \Phi^+(t, y)]_+ = \delta(x - y) \]

Calculate 2-point, time-ordered 2-time Green function:

\[ G(x, y) = \langle \mathcal{T} \Phi^+(x) \Phi(y) \rangle; \quad x = (t, x) \]

contains single-particle densities

\[ n(t, p) = \int dr \ G(t, r; t, 0) \ e^{ipr} \]
2PI Effective Action (Ψ-Functional)

\[ \Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr}(\ln G^{-1} + G_{0}^{-1}(\phi)G) + \]

Only 2-particle irreducible (2PI) loop diagrams

Conservation laws fulfilled
Equilibration of the 1D Bose gas

Number of particles $n(t,p)$ with momentum $p$:

\[ [TG, J. Berges, M. Seco \\ & M.G.Schmidt, PRA 72 (05); \\ J. Berges & TG, PRA 76 (07)] \]

- initial state: $^{23}$Na atoms in 1D, $n_1 = 10^7 \text{ m}^{-1}$
- interaction parameter $\gamma = \lambda m / (\hbar^2 n_1) = 7.5 \cdot 10^{-4}$
Equilibration of a 1D Bose gas

No. of particles $n(t, p_i)$ with momentum $p_i$:

\[ [TG, J. Berges, M.G. Schmidt, and M. Seco, PRA 72 (05); J. Berges & TG, PRA 76 (07)] \]
Equilibration of a 1D Bose gas

No. of particles \( n(t,p_i) \) with momentum \( p_i \):

\[
\frac{n(t,p_i)}{n_L} = \left( \exp\left( \frac{\omega(p) - \mu}{k_B T} \right) - 1 \right)^{-1}
\]

Far from equilibrium (no fluct.-dissip. rel.)
Near equilibrium (fluct.-dissip. rel.)

[TG, J. Berges, M.G. Schmidt, and M. Seco, PRA 72 (05); J. Berges & TG, PRA 76 (07)]
Nonthermal Fixed Points
Equilibration

Transient, metastable state
e.g. Turbulence
Non-thermal fixed point
Non-thermal fixed point

\[ n(t, p) \sim p^{-\alpha} \]

initial conditions
\[ n(t=0, p) \]

thermal equilibrium
\[ n_{BE}(p) \]

[Fig. courtesy: J. Berges '08]
Wave Turbulence – e.g. on water

Theory prediction:

$$E_\omega \sim \omega^{-17/6}.$$

[Zakharov & Filonenko (67)]
Wave turbulence

Stationary scaling \( n(k) \) within \textbf{inertial} region:

\[
\log n(k) \sim k^{-\zeta}
\]
Movie 1: Phase evolution & Spectrum

\[ \Psi(\mathbf{\rho}, t) = \sqrt{n(\mathbf{\rho}, t)} \exp[i\varphi(\mathbf{\rho}, t)] \]

\[ n(k) = \langle \Psi^*(\mathbf{k})\Psi(\mathbf{k}) \rangle_{\text{angle average}} \]

http://www.thphys.uni-heidelberg.de/~smp/gasenzer/videos/boseqt.html
Spectrum in 2+1 D

B. Nowak, D. Sexty, TG, PRB 84: 020506(R), 2011
Cascades in 2+1 D

![Graphs showing occupation number $n(k)$ vs. radial momentum $k$.](image)
Imagine you had a balance equation for the radial flux

\[ \partial_t n(k) = - \partial_k Q(k) \]
Transport in momentum space

Transport equation (Quantum Boltzmann eq.):

\[ \partial_t n(k) = - \partial_k Q(k) \sim k^{d-1} J(k) \]

\[ = k^{d-1} d\Omega_k \int d^dp d^dq d^dr |T_{kpqr}|^2 \delta(k + p - q - r) \delta(\omega_k + \omega_p - \omega_q - \omega_r) \]

\[ \times [(n_k + 1)(n_p + 1)n_q n_r - n_k n_p (n_q + 1)(n_r + 1)] \]

in-scattering rate

out-scattering rate

dilute Bose gas: \[ T_{kpqr} \equiv g = \frac{4\pi a_0}{m} = \text{const.} \]
Transport in momentum space

Radial transport equation (Quantum Boltzmann):

\[ \partial_t n(k) = - \partial_k Q(k) \sim k^{d-1} J(k) \]

\[ = k^{d-1} d\Omega_k \int \frac{d^d p \, d^d q \, d^d r}{|T_{k p q r}|^2} \delta(k + p - q - r) \delta(\omega_k + \omega_p - \omega_q - \omega_r) \]

\[ \times [(n_k + 1)(n_p + 1)n_q n_r - n_k n_p (n_q + 1)(n_r + 1)] \]

Stationary distribution \( n(k, t) \equiv n(k) \) if \( Q(k) \equiv Q \)

This requires a particular scaling of \( n(k) \sim k^{-\zeta} \)
Transport in momentum space

Quantum Boltzmann breaks down for large $n$, once $|T_{kpqr}|n_k \gg O(1)$

\[
\frac{\partial}{\partial t} n(k) = -\partial_k Q(k) \sim k^{d-1} J(k)
\]

\[
= k^{d-1} d\Omega_k \int d^d p \, d^d q \, d^d r \, |T_{kpqr}|^2 \delta(k + p - q - r) \delta(\omega_k + \omega_p - \omega_q - \omega_r)
\]

\[
\times [(n_k + 1)(n_p + 1)n_qn_r - n_kn_p(n_q + 1)(n_r + 1)]
\]

here: $T_{kpqr} \equiv g = \text{const.}$

Cured by effective many-body T-Matrix:

\[
|T|^2 = g^2 \rightarrow |T_{k^MB}|^2 \sim \frac{g^2}{1 + (gkn_k)^2}
\]
Dyn. QFT: Resummed Vertex

\[ p = (p_0, p) : \]

\[ J(p) := \sum_{ab}^\rho (p) F_{ba} (p) - \sum_{ab}^F (p) \rho_{ba} (p) \doteq 0 \]

\[ \Sigma_{ab} (x,y) = \]

Vertex bubble resummation: (e.g. 2PI to NLO in $1/N$)

[Dynamics: J. Berges, (02); G. Aarts et al., (02); Nonthermal fixed points: J. Berges, A. Rothkopf, J. Schmidt, PRL (08)]
Bose gas in $d$ spatial dimensions $n \sim k^{-\zeta}$

New exponent beyond Quantum Boltzmann!

$\zeta = d + 2$

$\zeta = d$

$n$ vs. momentum $k$

C. Scheppach, J. Berges, TG PRA 81 (10) 033611
Solitons in 1 spatial dimension
Solitons in 1 spatial dimension
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