Discrete Holomorphicity in the Chiral Potts Model

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February 18, 2015

Abstract

We construct lattice parafermions for the $Z(N)$ chiral Potts model in terms of quasi-local currents of the underlying quantum group. We show that the conservation of the quantum group currents leads to twisted discrete-holomorphicity (DH) conditions for the parafermions. At the critical Fateev-Zamolodchikov point the parafermions are the usual ones, and the DH conditions coincide with those found previously by Rajabpour and Cardy. Away from the critical point, we show that our twisted DH conditions can be understood as deformed lattice current conservation conditions for an underlying perturbed conformal field theory in both the general $N \geq 3$ and $N = 2$ Ising cases.

1 Introduction

The chiral Potts model is a two-dimensional statistical model defined by spin variables subject to a $\mathbb{Z}_N$-symmetric, local interaction, and was introduced in the 1980s as a lattice model for commensurate-incommensurate phase transitions [1]. A few years later, it became of great interest for mathematical physics as a solution of the star-triangle equations [2] in which the Boltzmann weights do not satisfy the difference property, and also as a superintegrable [3] generalisation of the Ising model. An important step in the understanding of the model was its identification [4] as a descendent of the well-studied six-vertex model – more precisely, the integrable chiral Potts model is based on $\mathbb{Z}_N$ cyclic representations [5, 6] of the affine quantum algebra $U_q(\widehat{\mathfrak{sl}}_2)$ (see also [7]). This class of representations was then studied in detail and generalised to other quantum algebras in [8–10].

The chiral Potts model displays very peculiar physical features. In the superintegrable regime, it describes a commensurate-incommensurate phase transition in an intrinsically anisotropic lattice model [11–14]. This behaviour is believed to appear also in the ordinary integrable regime, where the chiral Potts model provides an integrable chiral deformation [15] of the Fateev-Zamolodchikov (FZ) clock model [16]. The latter is an integrable $\mathbb{Z}_N$-symmetric spin model whose scaling limit is described by the $\mathbb{Z}_N$-parafermionic current algebra [17], an extension of the Virasoro algebra generating the spectrum of a Conformal Field Theory (CFT). By a simple

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inspection of the Boltzmann weights, Rajabpour and Cardy have observed [18] that one of the parafermionic currents $\{\psi_k\}$ of the $\mathbb{Z}_N$-parafermionic CFT has a lattice analog in the FZ clock model, which is a *discretely holomorphic* operator, i.e., it satisfies a discrete version of the Cauchy-Riemann equations. Discretely holomorphic parafermions have been found empirically in a number of critical lattice models [18–21], and are a crucial ingredient to the rigorous study of bulk [22–26] and boundary [27] critical properties. In a recent paper [28], we have shown that the origin of discretely holomorphic parafermions for loop models can be traced to the underlying quantum algebraic structure, following the construction of quasi-local conserved currents by Bernard and Felder [29].

The object of the present paper is to use the integrability and $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry of the chiral Potts model to (i) explain the algebraic origin of the discretely holomorphic parafermions observed [18] for the FZ clock model, and (ii) extend the discrete Cauchy-Riemann equations to the chiral regime around the FZ point, and analyse their physical meaning in the scaling limit.

The paper is organised as follows. Sections 2, 3 and 4 are devoted to reviewing some background material, respectively on the basics of the chiral Potts model, the Bernard-Felder construction, and the $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry of the chiral Potts model. In Section 5 we construct the quasi-local operators associated to generators of $U_q(\widehat{\mathfrak{sl}}_2)$ (which reduce to the lattice parafermions of [18] at the FZ point), and we give an explicit form of the linear relations generalising the discrete Cauchy-Riemann equations for these operators. In Section 6 we interpret these linear relations in terms of perturbed CFT [30] and discuss both the general $N \geq 3$ model and $N = 2$ Ising case. Finally, we summarise our findings in Section 7.

2 The Chiral Potts Model

2.1 Definitions

The chiral Potts (CP) model [1–3] is a statistical model where the variables are spins $a_j \in \mathbb{Z}_N$ living on the sites of a square lattice $\mathcal{L}$. The Boltzmann weight of a spin configuration $\{a_j\}$ is invariant under a rotation of all spins ($a_j \to a_j + 1 \mod N$), and is specified by the discrete functions $W_{\langle ij\rangle}$ associated to the edges of $\mathcal{L}$:

$$\mathcal{W}[\{a_j\}] = \prod_{\langle ij\rangle} W_{\langle ij\rangle}(a_i - a_j)\,, \tag{2.1}$$

where $\langle ij\rangle$ denotes a pair of neighbouring sites, connected by an oriented edge $i \to j$. Let us describe the specific choice of weight functions $W_{\langle ij\rangle}$ which renders the model integrable. In addition to the number of spin states N, we fix an external real parameter $k' \geq 0$. Each rapidity line carries a spectral parameter ξ given as a triplet of complex numbers $\xi = (x, y, \mu)$ obeying the algebraic equations

$$x^N + y^N = k(1 + x^N y^N) \qquad \text{and} \qquad \mu^N(1 - kx^N) = k'\,, \tag{2.2}$$

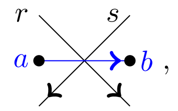
where we have set $k = \sqrt{1 - k'^2}$. A SW→NE (resp. NW→SE) edge crossed by rapidity lines (r, s) is assigned the weight function W_{rs} (resp. $\overline{W}_{rs}$), defined[1] for $a \in \{0, 1, 2, \dots\}$ as

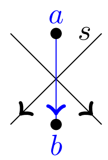
$$W_{rs}(a) = \left(\frac{\mu_r}{\mu_s}\right)^a \times \prod_{\ell=1}^{a} \frac{y_s - x_r\omega^\ell}{y_r - x_s\omega^\ell}\,, \; \overline{W}_{rs}(a) = (\mu_r\mu_s)^a \times \prod_{\ell=1}^{a} \frac{x_r\omega - x_s\omega^\ell}{y_s - y_r\omega^\ell}\,, \tag{2.3}$$

where $\omega = \exp(2i\pi/N)$, and we have used $\xi_r = (x_r, y_r, \mu_r)$ and $\xi_s = (x_s, y_s, \mu_s)$ to denote the spectral parameters attached to the rapidity lines r and s, respectively.

[1] The conditions (2.2) ensure that W_{rs} and $\overline{W}_{rs}$ are well-defined, i.e., $W_{rs}(a+N) = W_{rs}(a)$ and $\overline{W}_{rs}(a+N) = \overline{W}_{rs}(a)$.

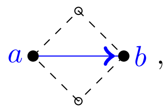
The CP weights are represented by

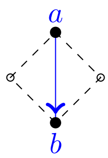
$$W_{rs}(a-b) = $$ 

, $$\overline{W}_{rs}(a-b) = $$ 

. (2.4)

We sometimes use an alternative graphical notation for CP weights that emphasises the fact that we can associate them with rhomboids on the covering lattice (i.e. the union of the CP lattice points, denoted by $\bullet$, and the dual lattice points denoted by by $\circ$). This notation is

$$W_{rs}(a-b) = $$ 

, $$\overline{W}_{rs}(a-b) = $$ 

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A homogeneous chiral Potts model partition function Z_{rs} is given as the sum over the height variables (i.e., the indices $a \in \{0, 1, \cdots, N-1\}$ at the positions marked by $\bullet$) associated with a lattice with spectral parameters ξ_r, ξ_s distributed as shown in Figure 1, in which all diagonal lines have downward arrows, which we have omitted for clarity.

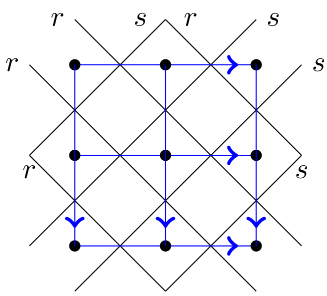


Figure 1: A homogenous Chiral Potts Lattice

The above weight functions satisfy [2] the star-triangle equations:

$$\sum_{d=0}^{N-1} \overline{W}_{rs}(a-d) W_{rt}(d-b) \overline{W}_{st}(d-c) = \rho_{rst} \times W_{rs}(c-b) \overline{W}_{rt}(a-c) W_{st}(a-b) \,, \tag{2.5}$$

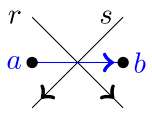
for any fixed spins (a, b, c), and the overall factor ρ_{rst} is a function of (ξ_r, ξ_s, ξ_t) .

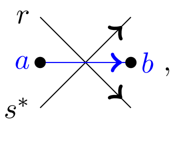
2.2 Crossing symmetry

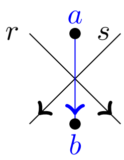
It is simple to see that the CP weights obey the crossing symmetry relations

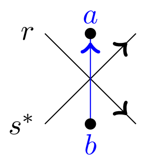
$$\begin{aligned} W_{r\,s}(a) &= \overline{W}_{s^*\,r}(a), \quad \overline{W}_{r\,s}(a) = W_{s^*\,r}(-a) \,, \\ \text{where} \quad (x, y, \mu)^* &= (\omega^{-1} y, x, 1/\mu), \end{aligned} \tag{2.6}$$

which can be indicated graphically by



$=$ 

, 

$=$ 

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2.3 Alternative parameterisation

The spectral parameter $\xi = (x, y, \mu)$ subject to the conditions (2.2) can be conveniently reparameterised as

$$x = e^{i(u+\phi)/N}\,, \qquad y = e^{i(u-\phi+\pi)/N}\,, \qquad \mu = e^{i(\bar{\phi}-\phi)/N}\,, \tag{2.7}$$

where the variables $(\phi, \bar{\phi}, u)$ are now related by

$$\sin\phi = -k\sin u\,, \qquad \sin\bar{\phi} = -\frac{ik}{k'}\cos u\,, \qquad \cos\phi = k'\ \cos\bar{\phi}\,. \tag{2.8}$$

Note that (2.8) amounts to two independent relations, so that one complex parameter among $(\phi, \bar{\phi}, u)$ remains free.

When approaching the self-dual line $\{\phi = \bar{\phi}, k' = 1\}$, it is convenient to scale u as

$$u = -i\log k + \frac{\pi}{2} + u'\,,$$

where u' is finite. In the limit $k' \to 1$, one then gets, on the self-dual line:

$$\sin\phi = \sin\bar{\phi} = -\frac{e^{-iu'}}{2}\,.$$

2.4 Transfer matrix and spin-chain Hamiltonian

In this paragraph, we consider a square lattice $\mathcal{L}$ tilted by 45°, so that the rapidity lines go along the horizontal and vertical directions, unlike in Figure 1. We denote by $\mathcal{T}_{rs}$ the transfer matrix comprising two horizontal rapidity lines with spectral parameter ξ_r, and $2L$ vertical rapidity lines with spectral parameter ξ_s, and we impose periodic boundary conditions in the horizontal direction. Then, as a consequence of the star-triangle equations (2.5), the transfer matrices with different values of the horizontal parameter commute:

$$[\mathcal{T}_{r_1,s}, \mathcal{T}_{r_2,s}] = 0\,. \tag{2.9}$$

Moreover, when $\xi_r = \xi_s$, the transfer matrix $\mathcal{T}_{rs}$ reduces to a cyclic translation e^{-iP}. Hence, one can define the associated Hamiltonian in the usual way in the limit $\xi_r \to \xi_s$, by writing:

$$\mathcal{T}_{rs} = e^{-iP} \times \left\{\mathbb{I} - (u_r - u_s)\mathcal{H}_s + O[(u_r-u_s)^2]\right\}\,. \tag{2.10}$$

This results in the spin chain Hamiltonian acting on the tensor space $V_L = \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$ (L times):

$$\begin{gathered}\mathcal{H}_s = \frac{1}{N\cos\bar{\phi}_s}\sum_{j=1}^{L}\sum_{n=1}^{N-1}\left[\bar{\alpha}_n (Z_j)^n + \alpha_n (X_j X_{j+1}^\dagger)^n\right]\,, \\ \alpha_n = \frac{\exp\left[i(2n-N)\phi_s/N\right]}{\sin(\pi n/N)}\,, \qquad \bar{\alpha}_n = k' \times \frac{\exp\left[i(2n-N)\bar{\phi}_s/N\right]}{\sin(\pi n/N)}\,.\end{gathered} \tag{2.11}$$

with periodic boundary conditions $X_{L+1} \equiv X_1$. The operators X_j and Z_j are defined through the elementary $N \times N$ matrices (X, Z) with coefficients

$$X_{ab} = \omega^a\ \delta_{ab}\,, \qquad Z_{ab} = \delta_{a,b-1}^{(\mathrm{mod}\ N)}\,, \tag{2.12}$$

which are characterised (up to a change of bases) by the relations $X^N = Z^N = \mathbb{I}$ and $ZX = \omega XZ$. The operator X_j (resp. Z_j) then acts as the matrix X (resp. Z) on the j-th factor of V_L, and as the identity matrix on the other factors. Note that (2.11) is Hermitian for any choice of real parameters $(\phi_s, \bar{\phi}_s, k')$.

2.5 $\mathbb{Z}_N$ charges and Kramers-Wannier duality

Let R be the global rotation of spins: $R = \prod_{j=1}^{L} Z_j$. Since R and $\mathcal{H}_s$ commute, we can diagonalise them simultaneously. We denote the eigenvalues of R as ω^{-m} with $m \in \mathbb{Z}_N$, and we refer to m as the $\mathbb{Z}_N$ charge. We may also consider the Hamiltonian (2.11) with "twisted" periodic boundary conditions $X_{L+1} \equiv \omega^{\overline{m}} X_1$, with $\overline{m} \in \mathbb{Z}_N$, and we call $\overline{m}$ the dual $\mathbb{Z}_N$ charge.

In the present context, Kramers-Wannier (KW) duality amounts to the following non-local change of bases. We introduce for $j \in \{1, \dots, L\}$, the dual operators

$$\overline{Z}_{j+1/2} = X_j X_{j+1}^\dagger \,, \qquad \overline{X}_{j+1/2} = \prod_{\ell=1}^{j} Z_\ell^\dagger \,, \tag{2.13}$$

and we set by convention $\overline{X}_{1/2} = \mathbb{I}$. We see readily that

$$\mathcal{H}_s = \sum_{j=1}^{L} \sum_{n=1}^{N-1} \left[\alpha_n (\overline{Z}_{j+1/2})^n + \bar{\alpha}_n (\overline{X}_{j-1/2} \overline{X}_{j+1/2}^\dagger)^n \right] \,, \tag{2.14}$$

with $\overline{X}_{j+1/2}^N = \overline{Z}_{j+1/2}^N = \mathbb{I}$, and $\overline{Z}_{j+1/2} \overline{X}_{\ell+1/2} = \omega^{\delta_{j\ell}} \overline{X}_{\ell+1/2} \overline{Z}_{j+1/2}$. Hence we recover the original form (2.11), except that the roles of α_n and $\bar{\alpha}_n$ have been exchanged. In terms of external parameters, KW duality acts as

$$(\phi, \bar{\phi}, k') \longrightarrow (\bar{\phi}, \phi, 1/k') \,. \tag{2.15}$$

Note that this transformation preserves the integrability condition $k' \cos\bar{\phi} = \cos\phi$. Let us examine its effects on the $\mathbb{Z}_N$ charges $(m, \overline{m})$. From the identities

$$\omega^m \, \mathbb{I} = \prod_{j=1}^{L} Z_j^\dagger = \overline{X}_{1/2}^\dagger \overline{X}_{L+1/2} \,, \qquad \omega^{\overline{m}} \, \mathbb{I} = X_1^\dagger X_{L+1} = \prod_{j=1}^{L} \overline{Z}_{j+1/2}^\dagger \,, \tag{2.16}$$

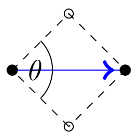
it is clear that KW duality exchanges the two charges:

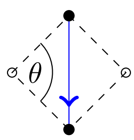
$$(m, \overline{m}) \longrightarrow (\overline{m}, m) \,. \tag{2.17}$$

2.6 The Fateev-Zamolodchikov case

When $\phi = \bar{\phi} = 0$ and $k' = 1$, the chiral Potts model reduces to the Fateev-Zamolodchikov (FZ) clock model [16]. In the scaling limit, the model is isotropic, and is described by the $\mathbb{Z}_N$-parafermionic CFT [17].

The weights of the FZ clock model enjoy the difference property, i.e., $W_{rs}(a)$ and $\overline{W}_{rs}(a)$ are functions of $u_s - u_r$ and a. Under these conditions, the star-triangle equations (2.5) are consistent with the following embedding of the model in the complex plane:

$W_{rs} =$ 

, $\quad \overline{W}_{rs} =$ 

, (2.18)

where the angle θ is defined as

$$\theta = u_s - u_r \,, \tag{2.19}$$

and the rhombi have a unit side length. Note that this choice also respects crossing symmetry, in that the embedding angle of the left hand side of the first crossing relation (2.6) is $u_s - u_r$, and the embedding angle of the right-hand-side is $\pi - u_s + u_r$ (and similarly for the second crossing relation).

2.7 The Ising case

The chiral Potts model with $N = 2$ corresponds to an Ising model with a partition function of the form

$$\mathcal{Z} = \sum_{\sigma_j=\pm 1} \prod_{\langle ij \rangle} \exp(K_{\langle ij \rangle}\sigma_i\sigma_j) \,, \tag{2.20}$$

where $K_{\langle ij \rangle} = K_1$ (resp. $K_{\langle ij \rangle} = K_2$) if $\langle ij \rangle$ is a horizontal (resp. vertical) edge. In this case, the algebraic relations (2.2) can be parametrised by Jacobi elliptic functions of modulus k [31]:

$$x = -\sqrt{k}\,\mathrm{sn}\beta \,, \qquad y = -\sqrt{k}\frac{\mathrm{cn}\beta}{\mathrm{dn}\beta} \,, \qquad \mu = \frac{\sqrt{k'}}{\mathrm{dn}\beta} \,. \tag{2.21}$$

For any value of k', the couplings are functions of $\beta_s - \beta_r$:

$$e^{-2K_1} = k'\mathrm{scd}(K - \beta_s + \beta_r) \,, \qquad e^{-2K_2} = k'\mathrm{scd}(\beta_s - \beta_r) \,, \tag{2.22}$$

where $\mathrm{scd}(u) = \mathrm{sn}\frac{u}{2}/(\mathrm{cn}\frac{u}{2}\,\mathrm{dn}\frac{u}{2})$, and K is the complete elliptic integral of the first kind of modulus k. Using elliptic identities, one gets the relation

$$\sinh 2K_1 \, \sinh 2K_2 = \frac{1}{k'} \,. \tag{2.23}$$

Expressions (2.22) show that the Ising model enjoys the difference property for any value of k. Requiring that the star-triangle relation can be represented as a geometric relation in the complex plane leads us to define the embedding angle (see Section 2.6) as

$$\theta_k = \frac{\pi}{K}(\beta_s - \beta_r) \,. \tag{2.24}$$

From (2.7) and (2.21), we have the relations

$$e^{iu} = -\frac{ik\,\mathrm{sn}\beta\,\mathrm{cn}\beta}{\mathrm{dn}\beta} \,, \qquad e^{i\phi} = \frac{i\,\mathrm{sn}\beta\,\mathrm{dn}\beta}{\mathrm{cn}\beta} \,, \qquad e^{i\bar{\phi}} = \frac{ik'\,\mathrm{sn}\beta}{\mathrm{cn}\beta\,\mathrm{dn}\beta} \,. \tag{2.25}$$

In the critical limit $k \to 0$, a way to recover the expression (2.19) for the embedding angle is to set

$$\beta = \frac{K}{\pi}\left(\frac{i}{2}\log p + 2\beta'\right) \,, \tag{2.26}$$

where β' is finite [p is the nome of elliptic functions in (2.21), with $k \sim 4p^{1/2}$]. In this regime, we have the expansions of elliptic functions (see [31]):

$$\begin{aligned} H(\beta) &= -ie^{i\beta'} + ie^{-i\beta'}p^{1/2} + O(p^{3/2}) \,, \\ H_1(\beta) &= e^{i\beta'} + e^{-i\beta'}p^{1/2} + O(p^{3/2}) \,, \\ \Theta(\beta) &= 1 - e^{2i\beta'}p^{1/2} + O(p^{3/2}) \,, \\ \Theta_1(\beta) &= 1 + e^{2i\beta'}p^{1/2} + O(p^{3/2}) \,. \end{aligned} \tag{2.27}$$

Using these, we find that $e^{iu} = -e^{2i\beta'} + O(p^{3/2})$, and hence (2.24) reduces to (2.19) up to corrections of order k^3.

2.8 Critical behaviour

Returning to the general $N \geq 3$ case, we shall describe the critical behaviour of the chiral Potts Hamiltonian (2.11) in the space of parameters $(\phi, \bar{\phi}, k')$.

On each of the self-dual (SD) lines

$$\mathrm{SD}_1 = \{\phi = \bar{\phi},\ k' = 1\} \qquad \text{and} \qquad \mathrm{SD}_2 = \{\phi = -\bar{\phi},\ k' = 1\}\,, \tag{2.28}$$

the model is invariant (modulo a global spin reversal) under KW duality, and thus it is massless. On each of the SD lines, for $\phi \neq 0$, the model is in an "incommensurate state", (i.e. the ground state is in a sector with non-zero momentum), and remains critical [15], with the same critical exponents as in the isotropic $\mathbb{Z}_N$-parafermionic CFT, but its correlations become anisotropic in Minkowski space. These SD lines correspond respectively to a perturbation of the $\mathbb{Z}_N$-parafermionic CFT by an operator of conformal spin $+1$ and -1.

As shown in [11–14], in the plane $\phi = \bar{\phi}$, the model remains massless in a limited region around SD_1, and then it undergoes a commensurate-incommensurate transition to a massive phase. It seems plausible that these results also hold outside this plane, i.e., there is a massless incommensurate phase around the plane $\{k' = 1\}$, and the model becomes massive for $|k' - 1|$ large enough. At the FZ point, the massless phase is "pinched", and any small perturbation outside the plane $k' = 1$ develops a finite mass.

3 Non-local Operators and Quantum Groups

In this section, we review the picture of non-local operators arising from quantum groups that was developed by Bernard and Felder in [29]. In [28], these currents were used in a direct way to construct discretely holomorphic operators in dense and dilute loop models. In the case of chiral Potts, we will show in Section 5 that the currents constructed using the method of [29] split naturally into two half-currents which obey a discrete holomorphicity condition.

The starting point of Bernard and Felder [29] is to consider a quasi-triangular Hopf algebra $\mathcal{A}$ (also known as a quantum group) defined in terms of a set of generators $\{J_a, \Theta_a{}^b, \widehat{\Theta}^a{}_b\}$, $a, b = 1, 2, \ldots, n$ that have the relations

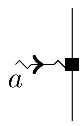
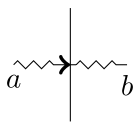
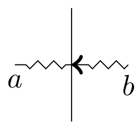
$$\Theta_a{}^b \widehat{\Theta}^c{}_b = \delta_{a,c} \quad \text{and} \quad \widehat{\Theta}^b{}_a \Theta_b{}^c = \delta_{a,c} \tag{3.1}$$

and the coproduct structure

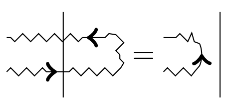
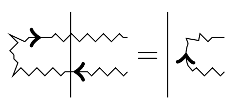
$$\Delta(J_a) = J_a \otimes 1 + \Theta_a{}^b \otimes J_b\,, \quad \Delta(\Theta_a{}^b) = \Theta_a{}^c \otimes \Theta_c{}^b\,, \quad \Delta(\widehat{\Theta}^a{}_b) = \widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b\,,$$

where $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ denotes the coproduct (for a gentle introduction to quantum groups see for example [32]). Note that we use the convention that repeated indices are summed over. The antipode and counit that complete the Hopf algebra structure can be found in [29].

It is helpful to introduce a graphical notation for representations of $\mathcal{A}$. We indicate a representation of $\mathcal{A}$ by a line and the action of the above generators of $\mathcal{A}$ on this representation by

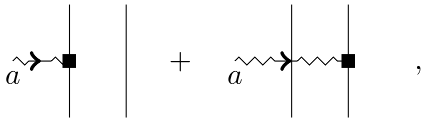
$J_a =$  , $\Theta_a{}^b =$  , $\widehat{\Theta}^a{}_b =$  .

Adopting the convention that composition $A \circ B$ means that B is above A, the inversion relations (3.1) are then represented as

 and  . (3.2)

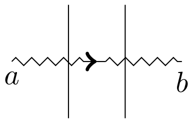
The action of the generators on tensor products of representations is indicated by

$\Delta(J_a) \;=$



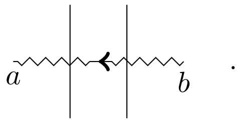
$J_a \otimes 1 \qquad\qquad \Theta_a{}^b \otimes J_b$

$\Delta(\Theta_a{}^b) \;=$



$\Theta_a{}^c \otimes \Theta_c{}^b$

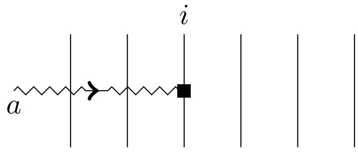
$\Delta(\widehat{\Theta}^a{}_b) =$



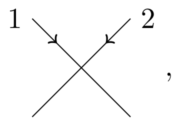
$\widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b$

Denoting the above coproduct by $\Delta^{(2)} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, it is then possible to define a coproduct $\Delta^{(L)} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (with L terms in the tensor product on the right) recursively by $\Delta^{(m+1)} = (\Delta \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots\cdots \mathbb{I})\Delta^{(m)}$. Then it follows that the representation of $\Delta^{(L)}(J_a)$ on an L-fold tensor product is indicated graphically by

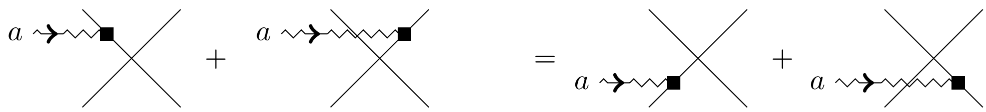
$$\Delta^{(L)}(J_a) = \sum_{i=1}^{L}$$



The above graphics becomes more useful when combined with the standard graphics for the R-matrix. The R-matrix of a quantum group, $\check{R} : V_1 \otimes V_2 \to V_2 \otimes V_1$, is a map between tensor products of representations V_1 and V_2 that commutes with the action of the quantum group $\mathcal{A}$. That is, we have $\check{R}\,\Delta(x) = \Delta(x)\,\check{R}$. We can represent the R-matrix graphically by



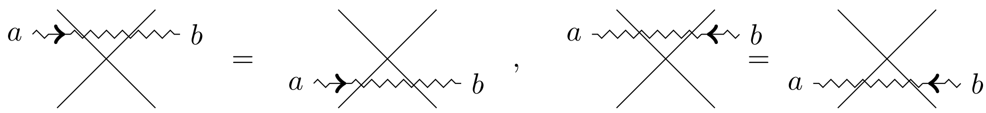
where the arrows serve to orient the picture and we view the R-matrix above as acting from top to bottom. We shall generally suppress these arrows. The commutation relations $\check{R}\,\Delta(x) = \Delta(x)\,\check{R}$ then have the following simple graphical realisations when $x = J_a, \Theta_a{}^b$, and $\widehat{\Theta}^a{}_b$:



(3.3)

$$\check{R}(J_a \otimes 1) \;+\; \check{R}(\Theta_a{}^b \otimes J_b) \;=\; (J_a \otimes 1)\check{R} \;+\; (\Theta_a{}^b \otimes J_b)\check{R},$$

and

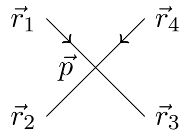


(3.4)

$$\check{R}(\Theta_a{}^c \otimes \Theta_c{}^b) \;=\; (\Theta_a{}^c \otimes \Theta_c{}^b)\check{R} \qquad \check{R}(\widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b) = (\widehat{\Theta}^a{}_c \otimes \widehat{\Theta}^c{}_b)\check{R}.$$

We now wish to define a non-local operator $j_a(x, y)$ associated with the insertion of the operator J_a at a point (x, y) in a 2D lattice model and an attached tail made up of tensor products of $\Theta_a{}^b$ and $\widehat{\Theta}^a{}_b$ along some path leading to a marked point on the boundary of the lattice. In order to do this, we again follow the approach of Bernard and Felder. Suppose we have a 2D lattice Λ consisting of 4-vertices at points $\vec{p} \in \mathbb{R}^2$. Let Λ' denote the lattice consisting of the points $\vec{r}$ which are the midpoints of the edges of Λ. Then if $V(\vec{r})$ denotes a $\mathcal{A}$ representation associated with the midpoint $\vec{r}$, the R-matrix associated with the vertex $\vec{p} \in \Lambda$ will be a map

$$\check{R}(\vec{p}) : V(\vec{r}_1) \otimes V(\vec{r}_4) \to V(\vec{r}_2) \otimes V(\vec{r}_3) \tag{3.5}$$



where the $\vec{r}_i \in \Lambda'$ are the indicated four midpoints surrounding the point $\vec{p}$ (some of the $V_{\vec{r}_i}$ will need to be isomorphic for the R-matrix to exist, which we assume to be the case). We can then define a vector space

$$V_\Lambda = \bigotimes_{\vec{r} \in \Lambda'} V(\vec{r}) \,,$$

and a linear operator $B : V_\Lambda \to V_\Lambda$, by

$$B = \bigotimes_{\vec{p} \in \Lambda} \check{R}(\vec{p}) \,.$$

The partition function of the vertex model with Boltzmann weights specified by the R-matrices is given by

$$\mathcal{Z} = \mathrm{Tr}_{V_\Lambda}(B) \,, \tag{3.6}$$

and the expectation value of any linear operator $\mathcal{O} : V_\Lambda \to V_\Lambda$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \mathrm{Tr}_{V_\Lambda}(\mathcal{O}B) \,. \tag{3.7}$$

The above expressions (3.6) and (3.7) may at first seem unusual – the partition function and correlation functions are more commonly given as the trace over a 1D tensor product space that would correspond to the Hilbert space in the quantum statistical mechanics interpretation of the partition function. However, it is a useful, and simple, exercise to check that the partition function does always reduce to the standard 1D trace. In contrast, the expectation value $\langle \mathcal{O} \rangle$ may be written as a 1D trace only in the special case when the operator $\mathcal{O}$ acts trivially at all $\vec{r}$ in V_Λ except those along some 1D line in the lattice.

In this paper, we are interested in operators $\mathcal{O}$ that act non-trivially on a set of midpoints $\vec{r}$ that will wind along a path γ from a marked interior point to a marked point on the boundary of the lattice. In this case the more general expression (3.7) will be required. To be more precise, we consider the operator $j_a^\gamma(\vec{r}) : V_\Lambda \to V_\Lambda$ constructed by the insertion of the representation of J_a on $V(\vec{r})$ at the midpoint $\vec{r}$, and the insertion of a "tail operator" constructed from the insertion of $\Theta_a{}^b$ or $\widehat{\Theta}^a{}_b$ along a line of midpoints specified by the path γ that terminates at some fixed, but arbitrary, point on the boundary of the lattice. An example is shown in Figure 3.

The commutation relations with the R-matrix expressed by (3.3) and (3.4) have two immediate consequences for expectation values $\langle j_a^\gamma(\vec{r}) \rangle$. The second relation (3.4) implies that the expectation value is independent of the path γ and will depend only upon the insertion point $\vec{r}$ and the fixed boundary point. Thus we will from now on drop the γ path superscript on the current. The other commutation relation (3.3) implies that when inserted into an expectation value we have

$$j_a(\vec{r}_1) - j_a(\vec{r}_2) - j_a(\vec{r}_3) + j_a(\vec{r}_4) = 0 \,, \tag{3.8}$$

where $\vec{r}_i$ are the four edge midpoints points surrounding any vertex – as indicated in (3.5). After embedding the lattice into the complex plane it is this relation (3.8) that was interpreted as a discrete holomorphicity relation in several examples in the paper [28].

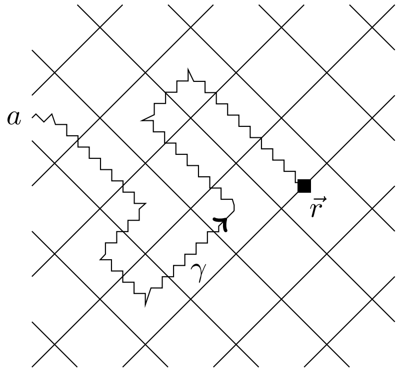


Figure 2: The insertion points and path of a non-local operator $j_a^\gamma(\vec{r})$.

4 Chiral Potts Weights and Representation Theory

In this section we review the construction of chiral Potts weights in terms of the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$ [8–10]. The notation we use is that of [9].

4.1 The quantum affine algebra $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$

We begin by defining $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$, which is an algebra over $\mathbb{C}$ generated by $e_i, f_i, t_i^{\pm 1}, z_i$ $(i = 0, 1)$, where $e_i, f_i, t_i^{\pm 1}$ satisfy the standard relation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, and z_0 and z_1 are two new central elements. The comultiplication of $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ is chosen as

$$\begin{aligned} \Delta(e_i) &= e_i \otimes \mathbb{I} + z_i t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + z_i^{-1} \otimes f_i, \\ \Delta(t_i) &= t_i \otimes t_i, \quad \Delta(z_i) = z_i \otimes z_i. \end{aligned}$$

The representations relevant to the chiral Potts occur when $q = -\exp(i\pi/N)$ where $N \in \{2, 3, 4, \cdots\}$, and we shall fix q to take this value from now on. We also define $\omega = q^2$. These representations are N-dimensional cyclic representations denoted $V_{rr'}$ and parametrised by a pair of points $(r, r') \in \mathcal{C}_k \times \mathcal{C}_k$. Here $\mathcal{C}_k$ is the algebraic curve (2.2) given by $(x, y, z) \in \mathbb{C}^3$ such that

$$x^N + y^N = k(1 + x^N y^N), \quad \mu^N = \frac{k'}{1 - kx^N} = \frac{1 - ky^N}{k'},$$

where $k^2 + k'^2 = 1$.

If $r = (x, y, z) \in \mathcal{C}_k$ and $r' = (x', y', z') \in \mathcal{C}_k$ then the representation $V_{rr'}$ is given by

$$\begin{aligned} &\pi_{rr'}(e_1) = \frac{q}{(q^2-1)^2}(x\mu\mu' Z - y')X, \qquad \pi_{rr'}(f_1) = \frac{c_0}{xx'\mu\mu'} X^{-1}(yZ^{-1} - x'\mu\mu'), \\ &\pi_{rr'}(t_1) = c_0 \mu\mu' Z, \ \pi_{rr'}(z_1) = c_0^{-1}, \\ &\pi_{rr'}(e_0) = \frac{q}{(q^2-1)^2} X^{-1}(y(\mu\mu')^{-1}Z^{-1} - x'), \quad \pi_{rr'}(f_0) = \left(\frac{c_0\mu\mu'}{x'} Z - \frac{q^2}{c_0 y}\right) X, \\ &\pi_{rr'}(t_0) = \frac{1}{c_0\mu\mu'} Z^{-1}, \ \pi_{rr'}(z_0) = c_0. \end{aligned} \tag{4.1}$$

Here, the objects X and Z are $N \times N$ matrices, such that $ZX = \omega XZ$ and $X^N = Z^N = \mathbb{I}$. In this paper, we shall fix X and Z as

$$X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \omega & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega^{N-1} \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The constant[2] c_0 also appearing in the above satisfies $c_0^2 = q^2 xx'/(yy')$.

4.2 The $\check{R}$-matrix

Consider now the R-matrix $\check{R}(rr', ss') : V_{rr'} \otimes V_{ss'} \to V_{ss'} \otimes V_{rr'}$ that obeys $\check{R}(rr', ss')\Delta(x) = \Delta(x)\check{R}(rr', ss')$ for $x \in \widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$. The approach of [9] starts with the Ansatz that this R-matrix is of the factorised form

$$\check{R}(rr', ss') = S_{r's}(T_{r's'} \otimes T_{rs})S_{rs'}\,, \tag{4.2}$$

where

$$S_{rs'} : V_{rr'} \otimes V_{ss'} \to V_{s'r'} \otimes V_{sr}\,, \quad T_{rs} : V_{sr} \to V_{rs}\,.$$

Then the relation

$$\check{R}(rr', ss')[\pi_{rr'} \otimes \pi_{ss'}(\Delta(x))] = [\pi_{ss'} \otimes \pi_{rr'}(\Delta(x))]\check{R}(rr', ss')$$

is ensured if S and T satisfy the stronger 'sufficiency conditions':

$$S_{rs'}[\pi_{rr'} \otimes \pi_{ss'}(\Delta(x))] = [\pi_{s'r'} \otimes \pi_{sr}(\Delta(x))]S_{rs'}\,, \tag{4.3}$$
$$(T_{r's'} \otimes 1)[\pi_{s'r'} \otimes \pi_{sr}(\Delta(x))] = [\pi_{r's'} \otimes \pi_{sr}(\Delta(x))](T_{r's'} \otimes 1)\,, \tag{4.4}$$
$$(1 \otimes T_{rs})[\pi_{r's'} \otimes \pi_{sr}(\Delta(x))] = [\pi_{r's'} \otimes \pi_{rs}(\Delta(x))](1 \otimes T_{rs})\,, \tag{4.5}$$
$$S_{r's}[\pi_{r's'} \otimes \pi_{rs}(\Delta(x))] = [\pi_{ss'} \otimes \pi_{rr'}(\Delta(x))]S_{r's}\,. \tag{4.6}$$

These sufficiency conditions were in turn found to be satisfied by the choice

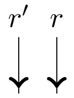
$$S_{rs}(v_{\varepsilon_1} \otimes v_{\varepsilon_2}) = W_{rs}(\varepsilon_1 - \varepsilon_2)(v_{\varepsilon_2} \otimes v_{\varepsilon_1})\,, \qquad T_{rs}v_\varepsilon = \sum_{a=0}^{N-1} \overline{W}_{rs}(a)v_{\varepsilon-a}\,,$$

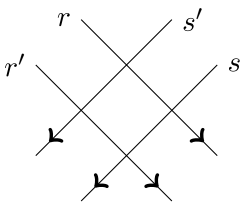
with the coefficients given by

$$\frac{W_{rs}(n)}{W_{rs}(n-1)} = \frac{\mu_r}{\mu_s}\frac{y_s - x_r\omega^n}{y_r - x_s\omega^n}\,, \qquad \frac{\overline{W}_{rs}(n)}{\overline{W}_{rs}(n-1)} = \mu_r\mu_s\frac{x_r\omega - x_s\omega^n}{y_s - y_r\omega^n}\,,$$

where we have now switched to the notation $a = (x_a, y_a, \mu_a)$ for a point $a \in \mathcal{C}_k$. These are the precisely the chiral Potts model Boltzmann weights (2.3) which can be found in [2]. Defining components $R(rr', ss')^{ab}_{cd}$ of the R-matrix by $\check{R}(rr', ss')(v_a \otimes v_b) = \sum_{c,d} R(rr', ss')^{ab}_{cd}(v_d \otimes v_c)$, then leads via (4.2) to the factorised expression

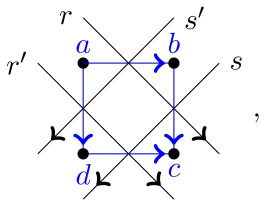
$$R(rr', ss')^{ab}_{cd} = W_{r's}(d-c)\overline{W}_{r's'}(a-d)\overline{W}_{rs}(b-c)W_{rs'}(a-b)\,. \tag{4.7}$$

It is useful to introduce a version of the standard 4-vertex graphical notation for R-matrices that is modified to deal with the representation $V_{rr'}$. We indicate the identity acting on the representation $V_{rr'}$ by a directed double line , and the R-matrix $\check{R}(rr', ss')$ by

$$\check{R}(rr', ss') =$$  ,

[2] The other constants appearing in Section 4 of [9] are here fixed as $c_1 = 1/c_0$ and $\kappa_i = 1/(q^2 - 1)$

The components $R(rr', ss')^{ab}_{cd}$ are then indicated by

$R(rr', ss')^{ab}_{cd} =$  ,

with the graphical conventions of Section 2.

5 Construction of non-local operators in the Chiral Potts model

In this section, we consider the non-local operators $j_a(\vec{r})$ discussed in Section 3 associated with the cyclic representations of $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ introduced in Section 4.

5.1 The current $j_{\bar{e}_0}$

Let us first consider the current associated with the generator $\bar{e}_0 := t_0 f_0$. We have the co-products

$$\Delta(\bar{e}_0) = \bar{e}_0 \otimes \mathbb{I} + t_0 z_0^{-1} \otimes \bar{e}_0 \,, \quad \text{and} \quad \Delta(t_0 z_0^{-1}) = t_0 z_0^{-1} \otimes t_0 z_0^{-1} \,, \tag{5.1}$$

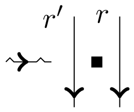
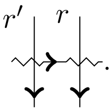
and thus $\bar{e}_0$ and $t_0 z_0^{-1}$ are respectively generators of type J_a and Θ_a^b in the notation of Section 3. It follows from (4.1) that the action of these generators on the representation $V_{rr'}$ is given by

$$\pi_{rr'}(\bar{e}_0) \;=\; X\left[x_{r'}^{-1} - y_r^{-1}\pi_{rr'}(t_0 z_0^{-1})\right] \,, \quad \pi_{rr'}(t_0 z_0^{-1}) = f_r f_{r'} Z^{-1} \,, \tag{5.2}$$
$$\text{where} \quad f_r \;:=\; \frac{y_r}{-q x_r \mu_r} \,.$$

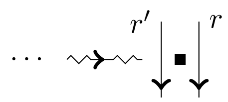
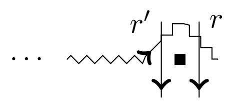
We now wish to follow the approach of Section 3 and consider the non-local operator $\bar{e}_0(x,t)$ associated with the insertion of the appropriate representation of the non-local operator

$$\cdots \otimes t_0 z_0^{-1} \otimes t_0 z_0^{-1} \otimes \bar{e}_0 \,. \tag{5.3}$$

The position (x,t) will correspond to a CP site (x,t) (indicated by a $\bullet$ in Figure 1) and we define $\bar{e}_0(x,t)$ such that the $\bar{e}_0$ in (5.3) acts on the representation associated with a pair of diagonal lines either side of the point (x,t). To make this definition more precise it is useful to modify the graphical notation of Section 3. To this end we introduce the following representation of the diagonal action of X and non-diagonal action of $\pi_{rr'}(t_0 z_0^{-1})$ on the representation V_{rr}:

$X \;\sim$  , $\pi_{(rr')}(t_0 z_0^{-1}) \;\sim$ .

It follows from (5.1) and (5.2) that the action of $\bar{e}_0(x,t)$ splits into two 'half-currents' which can be represented graphically as

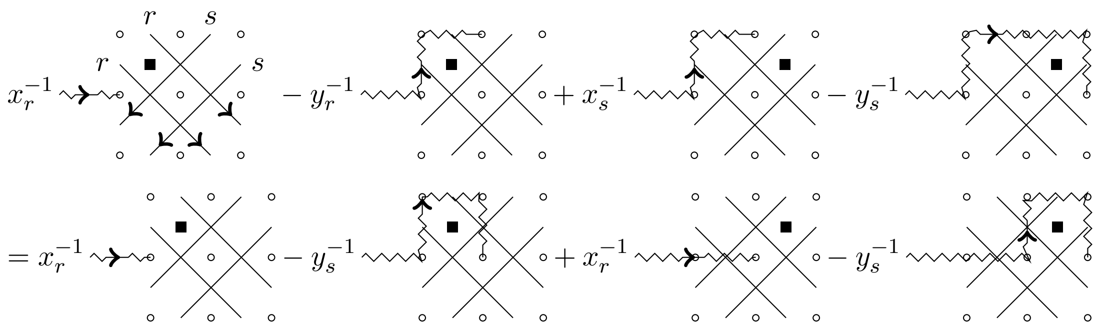
$\bar{e}_0(x,t) = x_{r'}^{-1}$  $-y_r^{-1}$  (5.4)

Here (x,t) is the CP site marked by a $\blacksquare$ and the left tail will wind through the rest of the CP lattice. Up to considerations of boundary conditions, which we will not need in this paper, the position of this tail is arbitrary due to the commutation of $t_0 z_0^{-1} \otimes t_0 z_0^{-1}$ with the R-matrix.

Let us now consider the sufficiency condition (4.3) in the case when $x = \bar{e}_0$, $r' = r$ and $s' = s$. Equation (4.3) becomes

$$S_{rs}[\pi_{rr} \otimes \pi_{ss}(\Delta(\bar{e}_0))] \quad = \quad [\pi_{sr} \otimes \pi_{sr}(\Delta(\bar{e}_0))]S_{rs}\,,$$

which can be represented graphically as



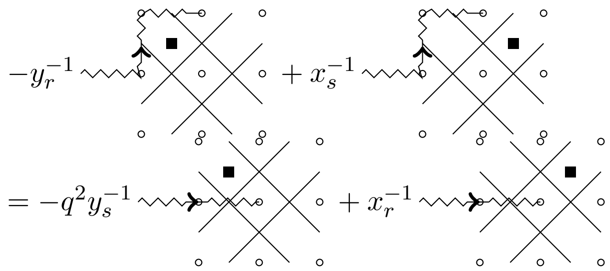
Several points are worth noting about this commutation relationship:

1. The tail operator is naturally associated with the edges of the dual CP lattice with vertices indicated by $\circ$.

2. We have appended a left horizontal tail to indicate that this relation may be embedded in the larger CP lattice with such a left tail.

3. There is cancellation of four of the terms.

After cancellation, we arrive at the four-term relation



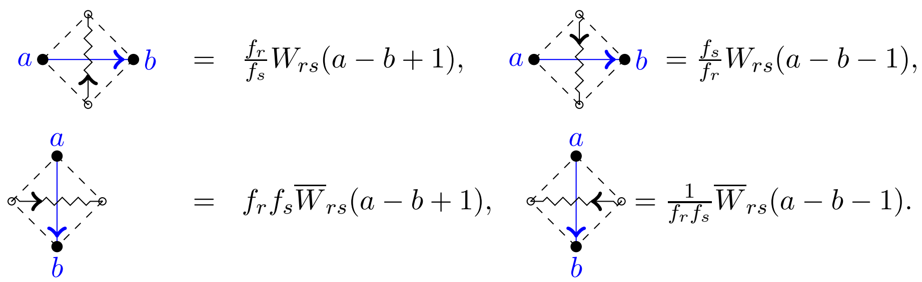
(5.5)

Note that we have used the relation $XZ^{-1} = q^2 Z^{-1} X$ to rewrite the first term on the right-hand-side of this equation.

In order to rewrite the relationship (5.5) in terms of CP weights, we need to understand the effect of the tail operator purely in terms of the modification of CP weights. Recall that the action of the tail operator $t_0 z_0^{-1}$ on the representation $V_{rr'}$ is given by

$$\pi_{rr'}(t_0 z_0^{-1}) = f_r f_{r'} Z^{-1}\,, \tag{5.6}$$

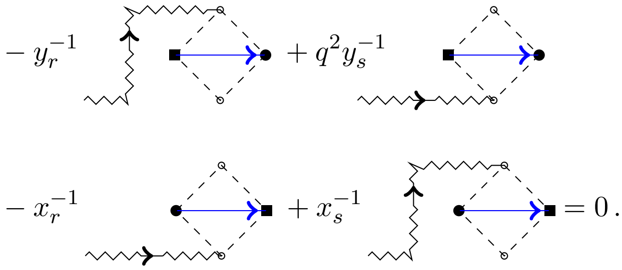
where Z^{-1} is the cyclic shift matrix that acts on canonical basis vectors v_a $(a = 0, 1, \cdots, N-1)$ as $Z^{-1} v_a = v_{a+1|_{\text{mod } N}}$. Hence, we can identify



(5.7)

In this way we identify the tail operator as a disorder operator for the CP model.

The diagrammatic identity of (5.5) can be written as a relation around any horizontal CP plaquette inserted into a larger partition function as



(5.8)

Consider the following non-local operator $j_{\bar{e}_0}(\vec{r})$ defined in terms of the above half-currents by

$$j_{\bar{e}_0}\left(\frac{\vec{r}_\sigma + \vec{r}_\mu}{2}\right) = T[\mu_{\bar{e}_0}(\vec{r}_\mu)\sigma(\vec{r}_\sigma)]\,,$$

where

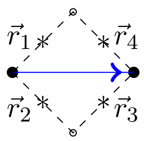
- $\sigma(\vec{r}_\sigma)$ corresponds to the insertion of $X = \blacksquare$ at embedded CP site $\vec{r}_\sigma$
- $\mu_{\bar{e}_0}(\vec{r}_\mu)$ is the above tail/disorder operator ending at embedded dual CP site $\vec{r}_\mu$
- T is 'tail ordering' defined as:

$$T[\mu_{\bar{e}_0}(\vec{r}_\mu)\sigma_1(\vec{r}_\sigma)] =$$
$$\text{quasi-local op. } \mu_{\bar{e}_0}(\vec{r}_\mu)\sigma(\vec{r}_\sigma) \text{ with tail} \begin{cases} \text{locally above } \vec{r}_\sigma \text{ if } \mathrm{Im}(z_\mu) \le \mathrm{Im}(z_\sigma) \\ \text{locally below } \vec{r}_\sigma \text{ if } \mathrm{Im}(z_\mu) > \mathrm{Im}(z_\sigma) \end{cases}$$

With this definition, the graphical relation (5.8) can be written simply as

$$-\,y_r^{-1} j_{\bar{e}_0}(\vec{r}_1) + q^2 y_s^{-1} j_{\bar{e}_0}(\vec{r}_2) - x_r^{-1} j_{\bar{e}_0}(\vec{r}_3) + x_s^{-1} j_{\bar{e}_0}(\vec{r}_4) = 0\,, \tag{5.9}$$

where we have denoted the mid-edges of the plaquette as follows:



. (5.10)

5.2 The operator $\mathcal{O}^{(s)}_{\bar{e}_0}$ and twisted Cauchy-Riemann equation

Let us parameterise (x, y, μ) in terms of $(u, \phi, \bar{\phi})$ as in (2.7), and introduce

$$\alpha_1 = \frac{u_s - u_r}{2} - \pi\,, \qquad \alpha_2 = -\frac{u_s - u_r}{2} + \pi\,, \qquad \alpha_3 = \frac{u_s - u_r}{2}\,, \qquad \alpha_4 = -\frac{u_s - u_r}{2}\,. \tag{5.11}$$

The linear relation (5.9) reads:

$$e^{i(\phi_r+\alpha_1)/N} j_{\bar{e}_0}(\vec{r}_1) - e^{i(\phi_s+\alpha_2)/N} j_{\bar{e}_0}(\vec{r}_2) + e^{i(-\phi_r+\alpha_3)/N} j_{\bar{e}_0}(\vec{r}_3) - e^{i(-\phi_s+\alpha_4)/N} j_{\bar{e}_0}(\vec{r}_4) = 0\,. \tag{5.12}$$

We choose $\theta = u_s - u_r$ as the embedding angle in (2.18), so that the quantity α_j coincides with the principal argument of $(z_\sigma - z_\mu)$ on the corresponding edge of the plaquette. Then the linear relation takes the form of a "twisted Cauchy-Riemann" relation:

$$e^{\frac{i\phi_r}{N}} \delta z_1\, \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_1) + e^{\frac{i\phi_s}{N}} \delta z_2\, \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_2) + e^{-\frac{i\phi_r}{N}} \delta z_3\, \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_3) + e^{-\frac{i\phi_s}{N}} \delta z_4\, \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_4) = 0\,, \tag{5.13}$$

where we have introduced the lattice current

$$\mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}) = \exp[-is\alpha(\vec{r})]\; j_{\bar{e}_0}(\vec{r})\,, \tag{5.14}$$

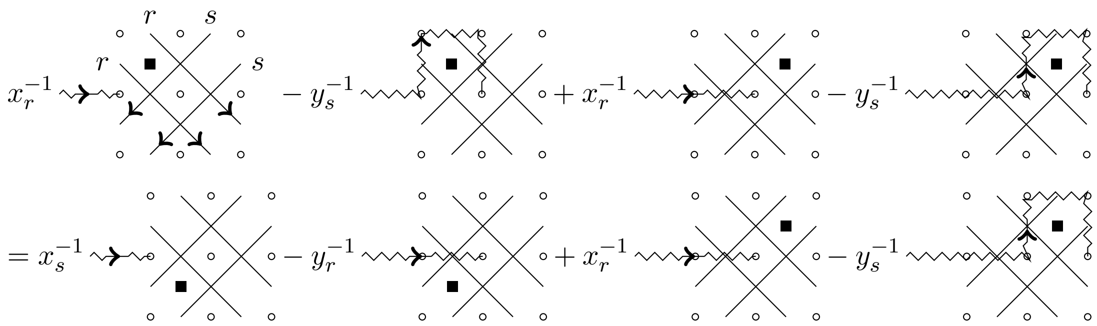
and where $\alpha(\vec{r})$ is the principal argument of the oriented edge $(z_\sigma - z_\mu)$ carrying the point $\vec{r}$, and the spin s is set to $s = 1 - 1/N$.

5.3 The $\overline{W}_{rs}$ plaquette

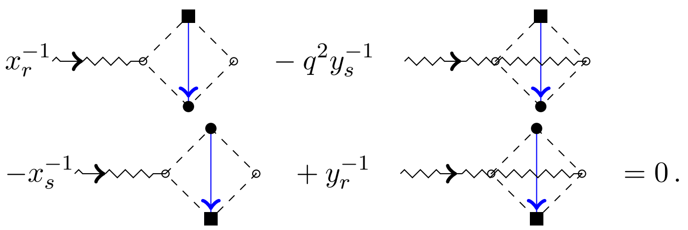
Equation (5.13) is a discrete integral relation around a W_{rs} plaquette. There is a direct extension of the above arguments that leads to a discrete integral relation for $\mathcal{O}_{\bar{e}_0}(z)$ around a $\overline{W}_{rs}$ plaquette. The starting point is to consider the commutation relation (4.5), that is

$$(T_{rs} \otimes 1)[\pi_{sr} \otimes \pi_{sr}(\Delta(\bar{e}_0))] \quad = \quad [\pi_{rs} \otimes \pi_{sr}(\Delta(\bar{e}_0))](T_{rs} \otimes 1)\,. \tag{5.15}$$

Using the splitting into half-currents given by (5.4), this can be represented by



Cancelling the common terms, and writing in terms of the $\overline{W}$ plaquette gives



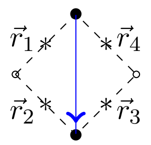
(5.16)

(5.17)

Defining the operator $\mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r})$ exactly as above, this leads to the discrete integral condition

$$e^{-\frac{i\phi_r}{N}}\delta z_1 \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_1) + e^{-\frac{i\phi_s}{N}}\delta z_2 \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_2) + e^{\frac{i\phi_r}{N}}\delta z_3 \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_3) + e^{\frac{i\phi_s}{N}}\delta z_4 \mathcal{O}^{(s)}_{\bar{e}_0}(\vec{r}_4) = 0\,, \tag{5.18}$$

around the embedded plaquette



. (5.19)

The coefficients appearing in (5.13) and (5.18) are such that when we consider any larger region, the contribution from internal points cancel and we are left with a discretely holomorphicity relation expressed solely on the boundary of the region.

5.4 Other currents

It is possible to define quasi-local operators in terms of the half-currents associated with the other generators. We consider these in turn.

5.4.1 The current for e_0

The coproduct and representation of e_0 are given by

$$\begin{aligned} \Delta(e_0) &= e_0 \otimes \mathbb{I} + t_0 z_0 \otimes e_0, \quad \Delta(t_0 z_0) = t_0 z_0 \otimes t_0 z_0\,, \\ \pi_{rr'}(e_0) &= \beta X^{-1}\left[x_{r'} - y_r \pi_{rr'}(t_0 z_0)\right], \quad \pi_{rr'}(t_0 z_0) = \frac{1}{\mu_r \mu_{r'}} Z^{-1}\,, \end{aligned} \tag{5.20}$$

where $\beta := -q/(q^2-1)^2$. Proceeding as above, may now define a new half-current

$$j_{e_0}((\vec{r}_\sigma + \vec{r}_\mu)/2) = T(\mu_{e_0}(\vec{r}_\mu)\sigma^\dagger(\vec{r}_\sigma)),$$

where

- $\sigma^\dagger(\vec{r}_\sigma)$ corresponds to the insertion of X^{-1} at embedded CP site $\vec{r}_\sigma$
- $\mu_{e_0}(\vec{r}_\mu)$ is a new disorder operator corresponding to the insertion of the operator $t_0 z_0$ along a path ending at the dual CP site $\vec{r}_\mu$. The effect of this disorder operator is similar to that of (5.7) except that $f_{r,s} \to \mu_{r,s}^{-1}$ on the right-hand side.

Defining $\mathcal{O}_{e_o}^{(-s)}(\vec{r}) = \exp[+is\alpha(\vec{r})] j_{e_0}(\vec{r})$, we can follow the previous analysis to arrive at the discrete antiholomorphicity conditions

$$e^{-\frac{i\phi_r}{N}}\delta\bar{z}_1\mathcal{O}_{e_o}^{(-s)}(\vec{r}_1) + e^{-\frac{i\phi_s}{N}}\delta\bar{z}_2\mathcal{O}_{e_o}^{(-s)}(\vec{r}_2) + e^{\frac{i\phi_r}{N}}\delta\bar{z}_3\mathcal{O}_{e_o}^{(-s)}(\vec{r}_3) + e^{\frac{i\phi_s}{N}}\delta\bar{z}_4\mathcal{O}_{e_o}^{(-s)}(\vec{r}_4) = 0\,,$$

$$e^{\frac{i\phi_r}{N}}\delta\bar{z}_1\mathcal{O}_{e_o}^{(-s)}(\vec{r}_1) + e^{\frac{i\phi_s}{N}}\delta\bar{z}_2\mathcal{O}_{e_o}^{(-s)}(\vec{r}_2) + e^{-\frac{i\phi_r}{N}}\delta\bar{z}_3\mathcal{O}_{e_o}^{(-s)}(\vec{r}_3) + e^{-\frac{i\phi_s}{N}}\delta\bar{z}_4\mathcal{O}_{e_o}^{(-s)}(\vec{r}_4) = 0\,,$$

around the W plaquette (5.10) and $\overline{W}$ plaquette (5.19) respectively.

5.4.2 The current for $\bar{e}_1$

A similar consideration leads us to define

$$j_{\bar{e}_1}((\vec{r}_\sigma + \vec{r}_\mu)/2) = T(\mu_{\bar{e}_1}(\vec{r}_\mu)\sigma^\dagger(\vec{r}_\sigma))\,,$$

where

- $\sigma^\dagger(\vec{r}_\sigma)$ corresponds to the insertion of X^{-1} at embedded CP site $\vec{r}_\sigma$
- $\mu_{\bar{e}_1}(\vec{r}_\mu)$ is a new disorder operator corresponding to the insertion of the operator $t_1 z_1^{-1}$ along a path ending at the dual CP site $\vec{r}_\mu$. Noting that

  $$\pi_{rr'}(t_1 z_1^{-1}) = \frac{1}{f_r f_r'} Z$$

  and comparing to equation (5.6), we see that this $\mu_{\bar{e}_1}(\vec{r}_\mu)$ disorder operator has the same action as in (5.7), but now with the arrow is directed outwards from the dual CP site $\vec{r}_\mu$.

Defining $\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}) = \exp[-is\alpha(\vec{r})] j_{\bar{e}_1}(\vec{r})$, we obtain the W and $\overline{W}$ discrete holomorphicity conditions

$$e^{-\frac{i\phi_r}{N}}\delta z_1\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_1) + e^{-\frac{i\phi_s}{N}}\delta z_2\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_2) + e^{\frac{i\phi_r}{N}}\delta z_3\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_3) + e^{\frac{i\phi_s}{N}}\delta z_4\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_4) = 0\,, \tag{5.21}$$

$$e^{\frac{i\phi_r}{N}}\delta z_1\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_1) + e^{\frac{i\phi_s}{N}}\delta z_2\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_2) + e^{-\frac{i\phi_r}{N}}\delta z_3\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_3) + e^{-\frac{i\phi_s}{N}}\delta z_4\mathcal{O}_{\bar{e}_1}^{(s)}(\vec{r}_4) = 0\,. \tag{5.22}$$

5.4.3 The current for e_1

Finally, by considering the half-currents that make up e_1, we arrive at the following definition of a quasi-local operator

$$j_{e_1}((\vec{r}_\sigma + \vec{r}_\mu)/2) = T(\mu_{e_1}(\vec{r}_\mu)\sigma(\vec{r}_\sigma))\,,$$

where

- $\sigma(\vec{r}_\sigma)$ corresponds to the insertion of X at embedded CP site $\vec{r}_\sigma$

- $\mu_{e_1}(\vec{r}_\mu)$ is a disorder operator corresponding to the insertion of the operator $t_1 z_1$ along a path ending at the dual CP site $\vec{r}_\mu$. This effect of the disorder operator is given by (5.7) with $f_{r,s} \to \mu_{r,s}^{-1}$ on the right-hand side and with the arrow leaving $\vec{r}_\mu$.

Defining $\mathcal{O}_{e_1}^{(-s)}(\vec{r}) = \exp[is\alpha(\vec{r})] j_{e_1}(\vec{r})$, we obtain the W and $\overline{W}$ discrete anti-holomorphicity conditions

$$e^{\frac{i\phi_r}{N}} \delta\bar{z}_1 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_1) + e^{\frac{i\phi_s}{N}} \delta\bar{z}_2 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_2) + e^{-\frac{i\phi_r}{N}} \delta\bar{z}_3 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_3) + e^{-\frac{i\phi_s}{N}} \delta\bar{z}_4 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_4) = 0\,,$$

$$e^{-\frac{i\phi_r}{N}} \delta\bar{z}_1 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_1) + e^{-\frac{i\phi_s}{N}} \delta\bar{z}_2 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_2) + e^{\frac{i\phi_r}{N}} \delta\bar{z}_3 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_3) + e^{\frac{i\phi_s}{N}} \delta\bar{z}_4 \mathcal{O}_{e_1}^{(-s)}(\vec{r}_4) = 0\,.$$

In summary, $\bar{e}_i$ yield half-currents with discrete holomorphicity and spin $s = (1 - 1/N)$, whilst e_i yield ones with discrete antiholomorphicity and spin $-s$. The coefficients in the discrete relations around either $(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4)$ plaquette are just $(e^{\pm i\phi_r/N}, e^{\pm i\phi_s/N}, e^{\mp i\phi_r/N}, e^{\mp i\phi_r/N})$ in all cases.

6 Physical interpretation

In this section, we discuss the physical meaning of the linear relations derived above for the operators $\mathcal{O}_{\bar{e}_i}^{(s)}$ and $\mathcal{O}_{e_i}^{(-s)}$. For simplicity, we restrict the discussion to the case of $\mathcal{O}_{\bar{e}_0}^{(s)}$ and the linear relation (5.13) around a W_{rs} plaquette, but very similar results hold for the other cases. Also, in order to lighten notation, we drop the indices $\bar{e}_0$ in $j_{\bar{e}_0}(\vec{r})$ or $\mathcal{O}_{\bar{e}_0}^{(s)}(\vec{r})$, and simply write $j(\vec{r})$ and $\mathcal{O}^{(s)}(\vec{r})$.

6.1 Discrete linear relation in the vicinity of the FZ point

At the FZ point $\phi_{r,s} = \bar{\phi}_{r,s} = 0$, (5.13) takes the simple form

$$\delta z_1 \mathcal{O}^{(s)}(\vec{r}_1) + \delta z_2 \mathcal{O}^{(s)}(\vec{r}_2) + \delta z_3 \mathcal{O}^{(s)}(\vec{r}_3) + \delta z_4 \mathcal{O}^{(s)}(\vec{r}_4) = 0\,. \tag{6.1}$$

Note that at this point $\mathcal{O}^{(s)}(z)$ coincides with the lattice parafermion of [18], and (6.1) is the discrete Cauchy-Riemann relation of the form $\bar{\partial}\psi_s = 0$ found empirically in [18].

In the vicinity of the FZ point, using (2.8), we can write the linearised relations

$$\phi_r = \cos\theta\ \phi_s + i\sin\theta\ \bar{\phi}_s + O(\phi_s^3, \bar{\phi}_s^3)\,, \qquad \bar{\phi}_r = i\sin\theta\ \phi_s + \cos\theta\ \bar{\phi}_s + O(\phi_s^3, \bar{\phi}_s^3)\,. \tag{6.2}$$

Hence, if we introduce the notation $\phi^\pm = (\phi \pm \bar{\phi})/2$, we get $\phi_r^\pm \sim e^{\pm i\theta}\phi_s^\pm$, and (5.13) becomes

$$\begin{aligned}
&\delta z_1 \mathcal{O}^{(s)}(\vec{r}_1) + \delta z_2 \mathcal{O}^{(s)}(\vec{r}_2) + \delta z_3 \mathcal{O}^{(s)}(\vec{r}_3) + \delta z_4 \mathcal{O}^{(s)}(\vec{r}_4) = \\
&\quad - \alpha_s^+ \left[t\mathcal{O}^{(s)}(\vec{r}_1) + t^{-1}\mathcal{O}^{(s)}(\vec{r}_2) + t\mathcal{O}^{(s)}(\vec{r}_3) + t^{-1}\mathcal{O}^{(s)}(\vec{r}_4) \right] \\
&\quad + \alpha_s^- \left[t^{-1}\mathcal{O}^{(s-2)}(\vec{r}_1) + t\mathcal{O}^{(s-2)}(\vec{r}_2) + t^{-1}\mathcal{O}^{(s-2)}(\vec{r}_3) + t\mathcal{O}^{(s-2)}(\vec{r}_4) \right] \\
&\quad + i(\alpha_s^+)^2 \left[t\mathcal{O}^{(s-1)}(\vec{r}_1) + t^{-1}\mathcal{O}^{(s-1)}(\vec{r}_2) + t\mathcal{O}^{(s-1)}(\vec{r}_3) + t^{-1}\mathcal{O}^{(s-1)}(\vec{r}_4) \right] \\
&\quad - i(\alpha_s^-)^2 \left[t^{-1}\mathcal{O}^{(s-1)}(\vec{r}_1) + t\mathcal{O}^{(s-1)}(\vec{r}_2) + t^{-1}\mathcal{O}^{(s-1)}(\vec{r}_3) + t\mathcal{O}^{(s-1)}(\vec{r}_4) \right] \\
&\quad + O((\alpha_s^\pm)^3),
\end{aligned} \tag{6.3}$$

where we have set

$$t = -ie^{i\theta}\,, \qquad \alpha_s^+ = \frac{e^{i\theta/2}\phi_s^+}{N}\,, \qquad \alpha_s^- = \frac{e^{-i\theta/2}\phi_s^-}{N}\,. \tag{6.4}$$

Note that the sum of coefficients multiplying the $\mathcal{O}$'s in each bracket on the RHS of (6.3) is $4\sin\theta$, which is proportional to the area of the plaquette. In the case $\theta = \pi/2$, then $t = 1$ and (6.3) takes the simpler form:

$$\delta z_1 \mathcal{O}^{(s)}(\vec{r}_1) + \delta z_2 \mathcal{O}^{(s)}(\vec{r}_2) + \delta z_3 \mathcal{O}^{(s)}(\vec{r}_3) + \delta z_4 \mathcal{O}^{(s)}(\vec{r}_4) = \\ - \alpha_s^+ \widehat{\mathcal{O}}^{(s)}(\vec{p}) + \alpha_s^- \widehat{\mathcal{O}}^{(s-2)}(\vec{p}) + i[(\alpha_s^+)^2 - (\alpha_s^-)^2]\widehat{\mathcal{O}}^{(s-1)}(\vec{p}) + O((\alpha_s^\pm)^3)\,, \qquad (6.5)$$

where we have defined $\vec{p}$ as the center of the plaquette, and

$$\widehat{\mathcal{O}}(\vec{p}) = \mathcal{O}(\vec{r}_1) + \mathcal{O}(\vec{r}_2) + \mathcal{O}(\vec{r}_3) + \mathcal{O}(\vec{r}_4)\,.$$

6.2 Comparison with perturbed CFT

In general, consider the perturbed action [30]

$$S = S_{\mathrm{CFT}} + g \int d^2r \;\Phi_{h,\bar{h}}(z,\bar{z})\,,$$

where S_{CFT} is the action of some CFT, and $\Phi_{h,\bar{h}}$ is an operator in the spectrum of this CFT. Suppose the unperturbed theory possesses a holomorphic current $\psi_s(z)$ with conformal spin s, and moreover, assume an OPE between the current and the perturbing field of the form:

$$\psi_s(z)\Phi_{h,\bar{h}}(w,\bar{w}) = \dots + \frac{\chi(w,\bar{w})}{z-w} + \dots \qquad (6.6)$$

By dimensional analysis, χ must have conformal weights $h_\chi = s + h - 1$ and $\bar{h}_\chi = \bar{h}$. In the perturbed theory, ψ_s is no longer a conserved current. More precisely, using the identity $\partial_{\bar{z}_1}[1/(z_1 - z_2)] = \pi\delta(\vec{r}_1 - \vec{r}_2)$, one gets at first order in g:

$$\bar{\partial}\psi_s(z,\bar{z}) = \pi g\; \chi(z,\bar{z})\,. \qquad (6.7)$$

So this simple line of arguments tells us how the conservation equation for the current ψ_s is modified by the perturbation.

Let us now specialise to the $\mathbb{Z}_N$ parafermionic CFT for $N \geq 3$ (the case $N = 2$ is treated separately in the next section). The chiral Potts model corresponds [15,33] to a perturbation of this CFT by the energy operator ε, together with the leading spin ± 1 operators:

$$S = S_{\mathrm{FZ}} + \int d^2r \,[\delta_+\Phi_+(z,\bar{z}) + \delta_-\Phi_-(z,\bar{z}) + \tau\varepsilon(z,\bar{z})]\;. \qquad (6.8)$$

The energy operator ε has dimensions $h_\varepsilon = \bar{h}_\varepsilon = 2/(N+2)$, and the spin ± 1 operators (which are actually the descendants $W_{-1}\varepsilon$ and $\overline{W}_{-1}\varepsilon$ in terms of the underlying W_N algebra) have dimensions $(h_{\Phi_+}, \bar{h}_{\Phi_+}) = (h_\varepsilon + 1, h_\varepsilon)$ and $(h_{\Phi_-}, \bar{h}_{\Phi_-}) = (h_\varepsilon, h_\varepsilon + 1)$. The parafermion of $\mathbb{Z}_N$ charges $m = \overline{m} = 1$ has conformal spin $s = 1 - 1/N$, and its conservation equation is modified by the perturbation according to (6.7):

$$\bar{\partial}\psi_s(z,\bar{z}) = \pi\delta_+\; \chi_+(z,\bar{z}) + \pi\delta_-\; \chi_-(z,\bar{z}) + \pi\tau\; \chi_0(z,\bar{z})\,, \qquad (6.9)$$

where the operators on the right-hand side have $\mathbb{Z}_N$ charges $m = \overline{m} = 1$, and conformal dimensions:

$$(h_{\chi_+}, \bar{h}_{\chi_+}) = (h_\varepsilon + s, \bar{h}_\varepsilon)\,, \quad (h_{\chi_-}, \bar{h}_{\chi_-}) = (h_\varepsilon + s - 1, \bar{h}_\varepsilon + 1)\,, \quad (h_{\chi_0}, \bar{h}_{\chi_0}) = (h_\varepsilon + s - 1, h_\varepsilon)\,.$$

Their conformal spins are thus s, $(s-2)$ and $(s-1)$, respectively.

Hence, we see that we can interpret (6.5) as a discrete version of the perturbed current conservation equation (6.9), with the parameters in the latter related by

$$\tau \propto (\delta_+^2 - \delta_-^2)\,. \qquad (6.10)$$

6.3 The Ising case

To make contact with previous work [19], it is convenient to introduce the "bare" current

$$J = Z \otimes Z \otimes \cdots \otimes Z \otimes X \,. \tag{6.11}$$

For this operator, the linear relations (5.13) and (5.21) arising from $j_{\bar{e}_0}$ and $j_{\bar{e}_1}$, when written in terms of the Ising parameterisation (2.24–2.25), read respectively:

$$\Theta_1(\beta_r)H(\beta_s)J(\vec{r}_1) + \Theta_1(\beta_s)H(\beta_r)J(\vec{r}_2) + \Theta(\beta_r)H_1(\beta_s)J(\vec{r}_3) - \Theta(\beta_s)H_1(\beta_r)J(\vec{r}_4) = 0 \,, \tag{6.12}$$

$$\Theta(\beta_r)H_1(\beta_s)J(\vec{r}_1) + \Theta(\beta_s)H_1(\beta_r)J(\vec{r}_2) - \Theta_1(\beta_r)H(\beta_s)J(\vec{r}_3) + \Theta_1(\beta_s)H(\beta_r)J(\vec{r}_4) = 0 \,. \tag{6.13}$$

When performing the $p \to 0$ expansion using (2.27), the terms of order $p^{1/2}$ vanish in the combination (6.12) $- i$(6.13), and we obtain:

$$\begin{aligned} &\delta z_1 \ \psi(\vec{r}_1) + \delta z_2 \ \psi(\vec{r}_2) + \delta z_3 \ \psi(\vec{r}_3) + \delta z_4 \ \psi(\vec{r}_4) = \\ &\qquad - ip \left[t^{-1}\bar{\psi}(\vec{r}_1) + t\bar{\psi}(\vec{r}_2) + t^{-1}\bar{\psi}(\vec{r}_3) + t\bar{\psi}(\vec{r}_4) \right] \,, \end{aligned} \tag{6.14}$$

where $t = -ie^{i\theta}$, and we have defined

$$\psi(\vec{r}) = e^{-i\alpha(\vec{r})/2} J(\vec{r}) \,, \qquad \bar{\psi}(\vec{r}) = e^{+i\alpha(\vec{r})/2} J(\vec{r}) \,. \tag{6.15}$$

In the RHS of (6.14), the sum of coefficients multiplying the $\bar{\psi}$'s is $(-4ip \sin\theta)$. Since the area of the plaquette is $\sin\theta$, the relation (6.14) is thus a discrete version of the massive Dirac equation with mass $m = 4p \sim k^2/4$:

$$\bar{\partial}\psi = -im \ \bar{\psi} \,. \tag{6.16}$$

Moreover, note that at $\theta = \pi/2$ we have $t = 1$, we recover the simple form of [19]:

$$\delta z_1 \ \psi(\vec{r}_1) + \delta z_2 \ \psi(\vec{r}_2) + \delta z_3 \ \psi(\vec{r}_3) + \delta z_4 \ \psi(\vec{r}_4) = -ip \left[\bar{\psi}(\vec{r}_1) + \bar{\psi}(\vec{r}_2) + \bar{\psi}(\vec{r}_3) + \bar{\psi}(\vec{r}_4) \right] \,. \tag{6.17}$$

Finally, if we use the linear relations for the e_0 and e_1 currents instead of $\bar{e}_0$ and $\bar{e}_1$, we find the second part of the Dirac equations, $\partial\bar{\psi} = im \ \psi$.

7 Conclusions

We have constructed the quasi-local operators associated to the $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry underlying the chiral Potts model, for any choice of integrable Boltzmann weights. The half-currents associated with these operators, when dressed with suitable local phase factors, satisfy "twisted" discrete Cauchy-Riemann equations (5.13) of the form

$$e^{i\phi_r/N}\delta z_1 \ \mathcal{O}(\vec{r}_1) + e^{i\phi_s/N}\delta z_2 \ \mathcal{O}(\vec{r}_2) e^{-i\phi_r/N}\delta z_3 \ \mathcal{O}(\vec{r}_3) + e^{-i\phi_s/N}\delta z_4 \ \mathcal{O}(\vec{r}_4) = 0 \,,$$

where ϕ_r and ϕ_s are the functions (2.7) of the spectral parameters r and s along the two directions of the lattice. At the isotropic critical point (FZ clock model), we have exhibited the algebraic origin of the lattice $\mathbb{Z}_N$-parafermions of [18]. In the generic case $N \geq 3$, and in the vicinity of the FZ point, we have shown that the above equation actually encodes (a discrete version of) the modified current conservation relation induced by a chiral perturbation of the $\mathbb{Z}_N$-parafermionic CFT. In the Ising case, this equation also allows us to recover the discrete massive Dirac equation of [19].

Thus, in the framework of the chiral Potts model, we have shown that the quantum group symmetry can be exploited to construct off-critical discrete parafermions and to probe the nature of the underlying perturbed conformal field theory.

Acknowledgments

The authors would like to thank John Cardy and Paul Zinn-Justin for useful discussions, and to acknowledge and thank Paul Fendley for explaining his published and unpublished work on parafermions [34, 35].

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