Quantum integrability in the lab

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NEWS AND PERSPECTIVES

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The anisotropic Heisenberg model is an interacting, one-dimensional system with quantum Hamiltonian

$$H = J \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta \left(S_i^z S_{i+1}^z - \tfrac{1}{4} \right) \right), \tag{1}$$

defined in terms of spin-$\frac{1}{2}$ quantum-mechanical spin operators $S_i^{x,y,z}$ that act on the ith site of a lattice. Theoretical physicists appreciate this model for two reasons: it is quantum integrable, and it is experimentally realized. In particular, in the antiferromagnetic regime $J > 0$, $\Delta > 1$ considered in [1], it accurately describes materials such as $CsCoCl_3$ and $CsCoBr_3$ [2, 3].

The model equation (1) is one of the most intensively studied of all quantum integrable models. With $\Delta = 1$ it becomes the Heisenberg or XXX model whose study dates from the 1931 paper of Bethe, in which the very first Bethe ansatz for the form of eigenvectors was formulated [4]. That the model is quantum integrable means that it is possible to compute some quantities such as eigenvectors and correlation functions exactly. Since the 1960s, an arsenal of sophisticated techniques has been developed for the study and solution of such quantum integrable systems and their close relatives, solvable lattice models. The technical tools involved include soliton theory, the Yang–Baxter equation and quantum inverse scattering (see for example [5]–[7] and references therein). These developments have in turn given rise to new areas of mathematics, most notably the field of quantum groups [8]. The humble model given by equation (1) has been one of the main testbeds in both the construction and application of these techniques. Despite this level of interest over many years, it was until recently still impossible to compute correlation functions beyond a few simple examples. This changed in the 1990s with the development of two complementary techniques for the exact computation of both correlation functions

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and form factors for this model. The first technique is usually called the vertex operator approach [9, 10]. The second makes use of the algebraic Bethe ansatz together with the solution of the quantum inverse problem (that is, a realization of local spin operators in terms of an operator called the monodromy matrix defined on the whole chain) [11, 12].

In the paper [1], the authors deploy both of these recently developed techniques in order to compute a quantity directly measurable in experiments. The quantity involved is the transverse dynamical structure factor (TDSF) $S^{-+}(k, w)$ defined by

$$S^{-+}(k, w) = \frac{1}{N} \sum_{j,j'=1}^{N} \mathrm{e}^{-\mathrm{i}k(j-j')} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}wt} \langle 0|S_j^-(t) S_{j'}^+(0)|0\rangle, \tag{2}$$

in which t denotes time and the matrix element is the vacuum expectation value, or equivalently the zero-temperature limit of the quantum statistical-mechanical trace. $S^{-+}(k, w)$ is measurable in inelastic neutron scattering experiments in which the neutrons lose energy w and momentum k.

The two techniques mentioned above can in principle directly compute each correlation function contributing to the sum in equation (2) (with the vertex operator approach restricted to the infinite N case). However, this straightforward approach is impractical, the problem being that the correlation function $\langle 0|S_j^-(0) S_{j'}^+(0)|0\rangle$ is given by a raw expression involving $|j - j'| + 1$ multiple integrals. It is therefore a hopeless task to compute the sum directly in this way. However, if the identity can be resolved in terms of energy eigenstates $|\alpha\rangle$ as $\mathbb{I} = \sum_\alpha |\alpha\rangle\langle\alpha|$, then by inserting this between $S_j^-(t)$ and $S_{j'}^-(0)$ in equation (2) we obtain

$$S^{-+}(p, w) = 2\pi \sum_{\alpha} |\langle 0|S_0^-(0)|\alpha\rangle|^2\, \delta(w - E_\alpha + E_0)\, \delta(p - p_\alpha). \tag{3}$$

Both of the techniques provide an exact realization of the spinon eigenstates of the Hamiltonian. These extended quasi-particle states are massive in the antiferromagnetic regime with a dispersion relation $E(p) = I\sqrt{1 - k^2 \cos^2(p)}$, where I and k are known functions of Δ. In the paper [1], the authors truncate the Lehmann sum in equation (3) at the two-spinon contribution.

The principal problems considered in [1] are the evaluation of the exact two-spinon form factor $\langle 0|S^-|\alpha\rangle$ using the two techniques, the precise understanding of the domain and the measure of the integral indicated by $\sum_\alpha$, the evaluation of the resulting contribution to the TDSF, and an estimation of the scale of the contribution to the complete TDSF (obtained by studying the percentage saturation of certain sum rules). The finite lattice form factors are extracted from existing results on the determinant representation of form factors and from the Bethe equation constraints for the allowed spinon momenta [11, 13]. The infinite lattice form factors are provided by the vertex operator approach to the XXZ model developed by the Kyoto School and summarized in the book [10]. The two-spinon contribution to the TDSF had been evaluated using the vertex operator approach in previous work [14], but the authors of [1] explain how and why their new results differ from the earlier ones.

The key observations of the paper [1] are that the TDSF results obtained from these two very different approaches agree for large N, and that they saturate different sum rules to a high degree over most of the momentum range in the first Brillouin zone. The

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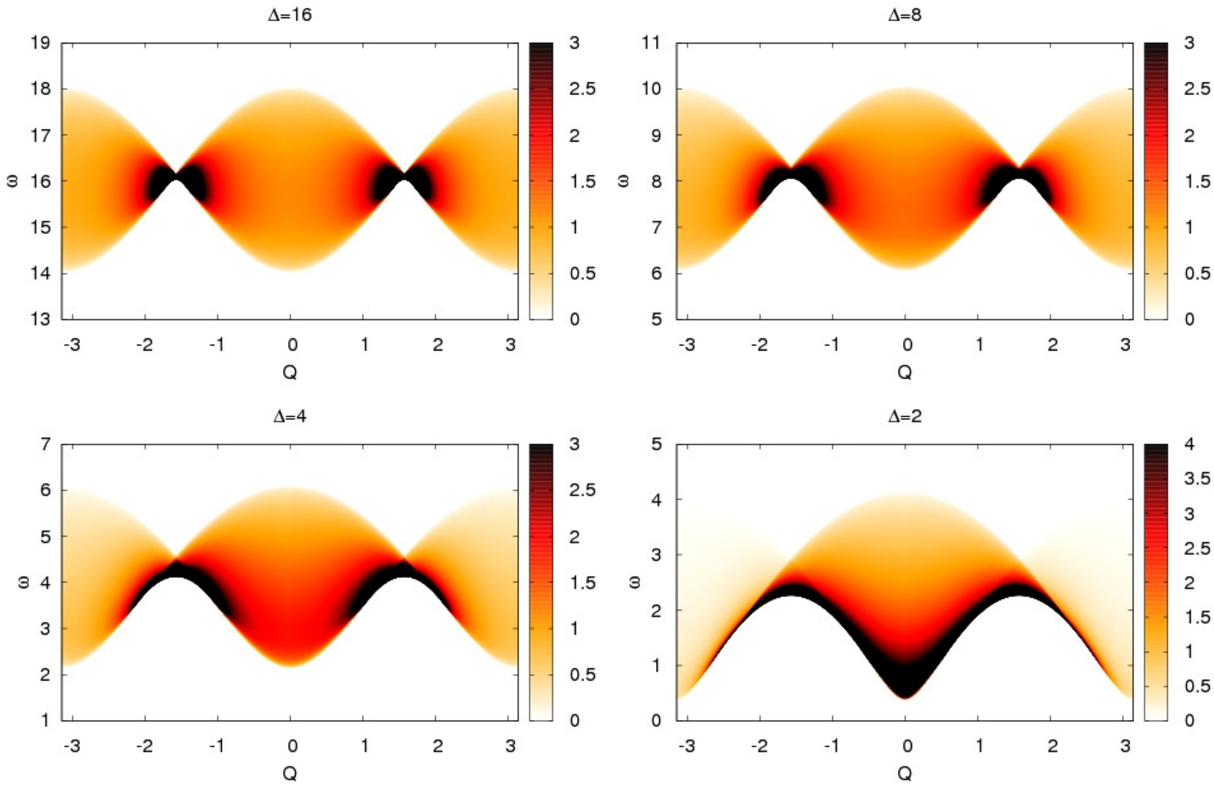

Figure 1. Results of [1] on the two-spinon contribution to the transverse dynamical structure factor $S^{-+}(Q, w)$ at zero field for $\Delta = 16$, 8, 4 and 2.

simplest sum rule that they consider is

$$\frac{1}{N} \sum_{k} \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{2\pi} S^{-+}(k, w) = \frac{1}{2}. \tag{4}$$

The results for the TDSF are reproduced from [1] in figure 1.

The paper [1] demonstrates in a compelling and elegant way that recent progress in integrable models has brought us to the point where we can compute dynamical quantities that are directly measurable in the laboratory. The Bethe ansatz and vertex operator approaches have also been developed for a wide range of other quantum integrable models beyond the antiferromagnetic XXZ model (see for example [15, 16]). Exact form factor and correlation functions for many of these models exist in the literature and are waiting to be exploited for real-world computations analogous to those of [1]. It is time to roll up our sleeves and get to work.

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