Spin Chains with Lattice Supersymmetry

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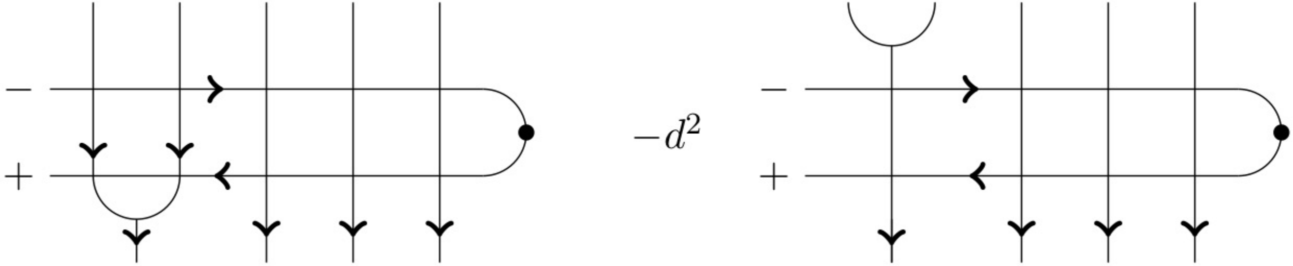

Plan

1. Introduction / Brief History
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3. An Algebraic Bethe Ansätz Analysis
4. Consequences of Above
5. The ground state – some combinatorial conjectures
6. Comments

Based on work with Junye Yang – arxiv: 1709.00442

1. Introduction / Brief History

This type of SUSY lattice model 1st constructed by

(F)endley, (S)choutens, de Boer (03);
F, (N)ienhuis, S (03); F, Yang (04)

- 1st paper mostly considered periodic 'hard-core fermion' model:

$$\{c_i, c_j^+\} = \delta_{ij} \quad , \quad \{c_i, c_j\} = \{c_i^+, c_j^+\} = 0$$

$$P_i = 1 - c_i^+ c_i \quad , \quad d_i^+ = P_{i-1} c_i^+ P_{i+1} \quad , \quad Q^+ = \sum_{i=1}^{L} d_i^+$$

$$\Rightarrow \quad Q^2 = Q^{+2} = 0 \quad , \quad H := \{Q, Q^+\}$$

- 2nd Mapped to XXZ:

$$\begin{tikzcd}
\cdots | \bullet | \cdots \arrow[r, mapsto] & \cdots \downarrow \cdots \\
\cdots | \quad | \cdots \arrow[u, "Q^+"] \arrow[r, mapsto] & \cdots \uparrow \uparrow \cdots \arrow[u, "Q_{XXZ}"]
\end{tikzcd}$$

fermion lattice size $L \longrightarrow$ XXZ lattice size $N = L - F$
& $Q_{XXZ} : V^{\otimes N} \rightarrow V^{\otimes N-1}$

- Survival of SUSY under this map subtle for periodic XXZ.
- Yang & F (04) considered cleaner case of open XXZ directly:

$$H^{(N)} = -\frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}\sigma_i^z\sigma_{i+1}^z\right) - \frac{1}{4}\left(\sigma_1^z+\sigma_N^z\right) + \frac{3N-1}{4}$$

NB (i) This is combinatorial point $q = e^{\pi i/3}$, $\Delta = -1/2$

(ii) **Not** $U_q(sl_2)$ inv. point $-\frac{(q-q^{-1})}{4}(\sigma_1^z - \sigma_N^z)$ [Pasquier/Saleur(90)]

- $H^{(N)} = Q^{\dagger(N)}Q^{(N)} + Q^{(N+1)}Q^{\dagger(N+1)}$

$Q^{(N)}: V^{\otimes N} \to V^{\otimes N-1}$; $Q^{\dagger(N)}: V^{\otimes N-1} \to V^{\otimes N}$

$$Q^{(N-1)}Q^{(N)} = 0 = Q^{\dagger(N+1)}Q^{\dagger(N)}$$

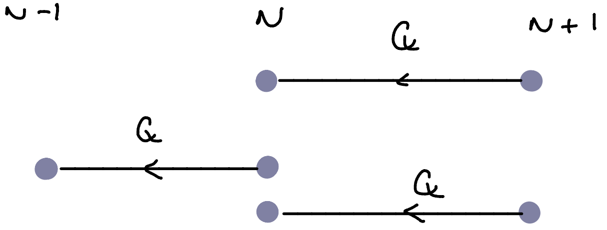
$$H^{(N-1)}Q^{(N)} = Q^{(N)}H^{(N)} \quad ; \quad H^{(N)}Q^{\dagger(N)} = Q^{\dagger(N)}H^{(N-1)}$$

- Why useful ?

(i) If $H|\Omega\rangle = E|\Omega\rangle$, then $E \geqslant 0$

$$\Big[\ \langle\Omega|H|\Omega\rangle = \langle\Omega|Q^+ Q|\Omega\rangle + \langle\Omega|QQ^+|\Omega\rangle$$
$$= \|Q|\Omega\rangle\|^2 + \|Q^+|\Omega\rangle\|^2\ \Big]$$

(ii) If $E > 0$, can arrange eigenstates into SUSY doublets



If $Q\chi = 0$

Then $H\chi = QQ^+\chi = E\chi$

So $\chi = \frac{QQ^+\chi}{E} \in \mathrm{Im}\, Q$

(iii) Only SUSY singlets have $E = 0$.

(v) Identify space of singlets as $\ker Q / \mathrm{Im}\, Q$.

If $\dim = 1$, then $\exists!$ g.s. with energy 0.

- In periodic case SUSY only in case:
  - (i) N odd, zero-momentum sector
  - (ii) N even, twist, zero-momentum sector.

  So SUSY singlets $\iff$ simple eigenvalue cases.

- Generalised and extended in many directions
  - M_k fermion models (adjacency > k not allowed)[F,N,S (03)]
  - XYZ model along comb. line [F, (H)agendorf (10, 11, 11)]
  - Higher spin s, $q = e^{i\pi/2(s+1)}$ [H (13)]
  - Cohomology class of open XXZ ground state [H, Liénardy (16)]
  - Modified fermionic models with fermion/hole symmetry [de Gier, Feher, N, Rusaczonek (16)]

⋮ ⋮

- What is missing is a detailed understanding of how SUSY fits into QISM / quantum group / reflection algebra / two-boundary TL algebra picture

  Our motivation .

- We

(i) Derived comm. relns between Q, Q^+ & generators of reflection alg.

ii) Obtained action on Bethe States .

2. The Open XXZ with SUSY

- Nilpotent ops: let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$

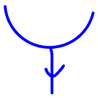
We want $Q^{(N)} : V^{\otimes N} \to V^{\otimes N-1}$; $Q^{(N)\dagger} : V^{\otimes N-1} \to V^{\otimes N}$

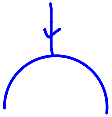
s.t. $Q^{(N-1)} Q^{(N)} = 0$; $Q^{(N+1)\dagger} Q^{(N)\dagger} = 0$

- Assume

$$Q^{(N)} = \sum_{i=1}^{N-1} (-1)^{i+1} q_{i,i+1} \quad ; \quad Q^{(N)\dagger} = \sum_{i=1}^{N-1} (-1)^{1+i} q_i^{\dagger}$$

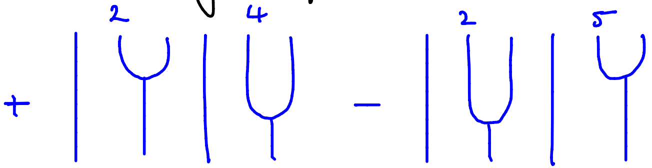
with

$q : V \otimes V \to V$ ~ 

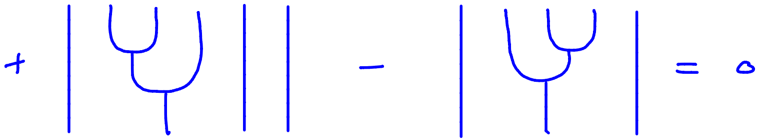
$q^{\dagger} : V \to V \otimes V$ ~ 

Nilpotency follows from (I) minus signs for non-local contribs.

e.g. $Q^{(N-1)} Q^{(N)}$ has contribs



II) Requiring



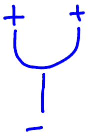
i.e. associativity $q(q\otimes \mathbb{I}) = q(\mathbb{I}\otimes q)$

(+ co-associativity $(q^+\otimes \mathbb{I})q^+ = (\mathbb{I}\otimes q^+)q^+$)

- The choice

$$q(v_{\varepsilon_1}\otimes v_{\varepsilon_2}) = \delta_{\varepsilon_1,+}\delta_{\varepsilon_2,+}\, v_- \quad ; \quad q^+ v_\varepsilon = \delta_{\varepsilon,-}(v_+\otimes v_+)$$

i.e only



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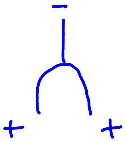
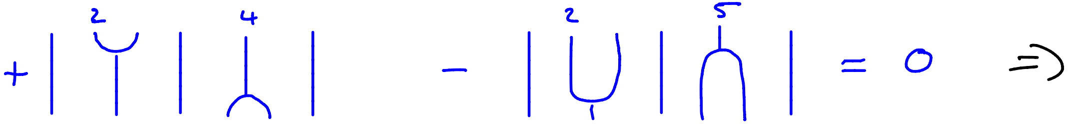
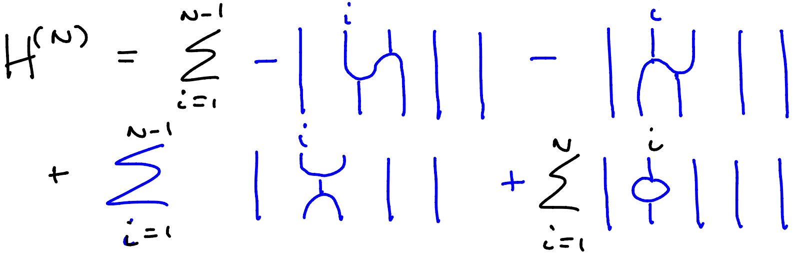


non-zero, satisfies these trivially.

- $H^{(N)} = Q^{(N)\dagger} Q^{(N)} + Q^{(N+1)} Q^{(N+1)\dagger}$ is local since





$$= -\frac{1}{2}\sum_{i=1}^{N-}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \frac{1}{2}\sigma_i^z\sigma_{i+1}^z\right) - \frac{1}{4}\left(\sigma_1^z + \sigma_N^z\right) + \frac{3N-1}{4}$$

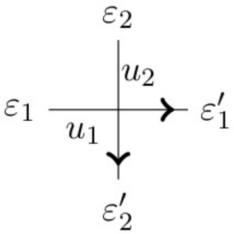
More general soln given in [Hagendorf & Liénardy (16)].
which gives non-diag. BCs.

3. An Algebraic Bethe Ansätz Analysis

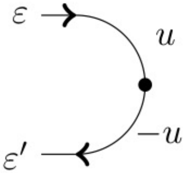
Q: How do $Q^{(N)}$, $Q^{(N)+}$ act on Bethe states?

- 1st Step: Sklyanin double-row transfer matrix picture.

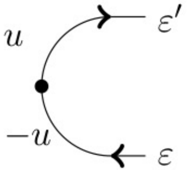
$R(u_1 - u_2)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'} =$



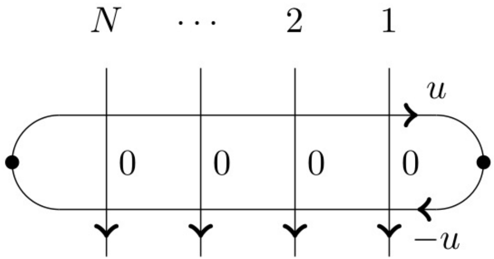
$K^-(u)^{\varepsilon}_{\varepsilon'} =$



$K^+(u)^{\varepsilon}_{\varepsilon'} =$



$t^{(N)}(u) =$



- Explicitly [Sklyanin 88]

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}$$

&

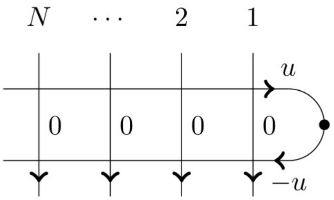
$$K^-(u) = \begin{pmatrix} d(u) & 0 \\ 0 & -a(u) \end{pmatrix}, \quad K^+(u) = K^-(u+\eta),$$

$$a(u) = \frac{\sinh(u+\eta)}{\sinh(\eta)}, \quad b(u) = \frac{\sinh(u)}{\sinh(\eta)}, \quad c(u) = 1, \quad d(u) = \frac{\sinh(u-\eta)}{\sinh(\eta)}.$$

$\eta = i\pi/3$ fixed

- $$t^{(N)\prime}(0) = -\frac{4i}{\sqrt{3}}\left[H^{(N)} + \left(\frac{1-N}{2}\right) \mathbb{1} \right]$$

- Define $U^{(N)}(u) =$



$$= \begin{pmatrix} \mathcal{A}^{(N)}(u) & \mathcal{B}^{(N)}(u) \\ \mathcal{C}^{(N)}(u) & \mathcal{D}^{(N)}(u) \end{pmatrix}$$

then
$$t^{(N)}(u) = \mathrm{Tr}_{V_0}\left(K_0^+(u)U^{(N)}(u)\right) = \frac{\sinh(u)}{\sinh(\eta)}\mathcal{A}^{(N)}(u) - \frac{\sinh(u+2\eta)}{\sinh(\eta)}\mathcal{D}^{(N)}(u),$$

$v_+ \otimes v_+ \otimes \cdots \otimes v_+$

- Bethe states $\mathcal{B}^{(N)}(\lambda_1)\mathcal{B}^{(N)}(\lambda_2)\cdots\mathcal{B}^{(N)}(\lambda_m)\,\Omega^{(N)}, \quad m \leq N,$

are 'on-shell' if
$$-\sinh(2\lambda_j)\frac{\Delta_+^{(N)}(\lambda_j)}{\Delta_-^{(N)}(\lambda_j)} = \prod_{k\neq j}\frac{\sinh(\lambda_j-\lambda_k+\eta)\sinh(\lambda_j+\lambda_k+2\eta)}{\sinh(\lambda_j-\lambda_k-\eta)\sinh(\lambda_j+\lambda_k)}$$

where

$$\mathcal{A}^{(N)}(\lambda)\Omega^{(N)} = \Delta_+^{(N)}(\lambda)\Omega^{(N)}, \quad \left[\mathcal{D}^{(N)}(\lambda)\sinh(2\lambda+\eta) - \mathcal{A}^{(N)}(\lambda)\sinh(\eta)\right]\Omega^{(N)} = \Delta_-^{(N)}(\lambda)\Omega^{(N)}.$$

Theorem: *We have the following commutation relations for $N \geq 2$:*

$$Q^{(N)}\mathcal{A}^{(N)} - d^2\mathcal{A}^{(N-1)}Q^{(N)} = (-1)^N c\, d\, E^+ \otimes \mathcal{B}^{(N-1)},$$

$$Q^{(N)}\mathcal{B}^{(N)} - d^2\mathcal{B}^{(N-1)}Q^{(N)} = 0,$$

$$Q^{(N)}\mathcal{C}^{(N)} - d^2\mathcal{C}^{(N-1)}Q^{(N)} = (-1)^N \Big(a\, c\, E^+ \otimes \mathcal{A}^{(N-1)} + c^2\, E^- \otimes \mathcal{B}^{(N-1)} + c\, d\, E^+ \otimes \mathcal{D}^{(N-1}$$

$$Q^{(N)}\mathcal{D}^{(N)} - d^2\mathcal{D}^{(N-1)}Q^{(N)} = (-1)^N b\, c\, E^+ \otimes \mathcal{B}^{(N-1)}.$$

where $E^{\varepsilon} : V \to \mathbb{C}$; $E_{\varepsilon_2}^{\varepsilon_1} : V \to V$; $E^{\varepsilon} v_{\varepsilon'} = \delta_{\varepsilon,\varepsilon'}$

$E_{\varepsilon''}^{\varepsilon} v_{\varepsilon'} = \delta_{\varepsilon,\varepsilon'} v_{\varepsilon''}$

Proof : We have both

$$Q^{(N+1)} = \sum_{i=1}^{N} (-1)^{1+i} q_{i,i+1} = \mathbb{I} \otimes Q^{(N)} + (-1)^{N+1} q_{N,N+1}, \text{ and}$$

$$\mathcal{A}^{(N+1)} = a^2 E_+^+ \otimes \mathcal{A}^{(N)} + b^2 E_-^- \otimes \mathcal{A}^{(N)} + acE_+^- \otimes \mathcal{B}^{(N)} + acE_-^+ \otimes \mathcal{C}^{(N)} + c^2 E_-^- \otimes \mathcal{D}^{(N)},$$

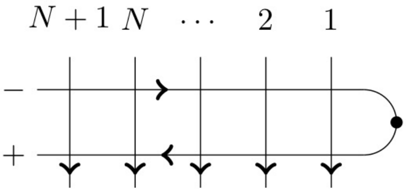
$$\mathcal{B}^{(N+1)} = bcE_-^+ \otimes \mathcal{A}^{(N)} + ab\,\mathbb{I} \otimes \mathcal{B}^{(N)} + bcE_-^+ \otimes \mathcal{D}^{(N)},$$

$$\mathcal{C}^{(N+1)} = bcE_+^- \otimes \mathcal{A}^{(N)} + ab\,\mathbb{I} \otimes \mathcal{C}^{(N)} + bcE_+^- \otimes \mathcal{D}^{(N)},$$

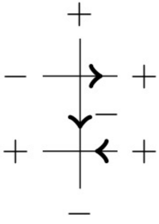
$$\mathcal{D}^{(N+1)} = c^2 E_+^+ \otimes \mathcal{A}^{(N)} + acE_+^- \otimes \mathcal{B}^{(N)} + acE_-^+ \otimes \mathcal{C}^{(N)} + b^2 E_+^+ \otimes \mathcal{D}^{(N)} + a^2 E_-^- \otimes \mathcal{D}^{(N)},$$

So we can use induction in N.

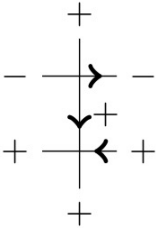
- e.g. $\mathcal{B}^{(N+1)} =$



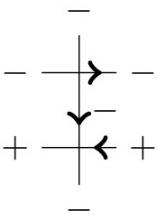
Non-zero weight configs around leftmost column are



bc



ab



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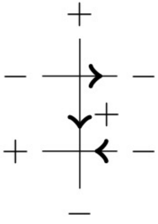


bc

Hence
$$\mathcal{B}^{(N+1)} = bcE_-^+ \otimes \mathcal{A}^{(N)} + ab\,\mathbb{I} \otimes \mathcal{B}^{(N)} + bcE_-^+ \otimes \mathcal{D}^{(N)}$$

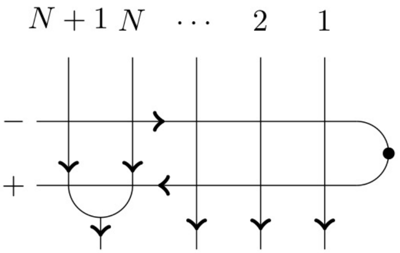
- If $[X]^{(N)} := Q^{(N)} X^{(N)} - d^2 X^{(N-1)} Q^{(N)}$, then

$$\begin{aligned}[\mathcal{B}]^{(N+1)} &= bc\,E_-^+ \otimes [\mathcal{A}]^{(N)} + ab\,\mathbb{I} \otimes [\mathcal{B}]^{(N)} + bc\,E_-^+ \otimes [\mathcal{D}]^{(N)} \\ &\quad + (-1)^{N+1} \left\{ q_{N+1,N}\, \mathcal{B}^{(N+1)} - d^2 \mathcal{B}^{(N)}\, q_{N+1,N} \right\}\end{aligned}$$

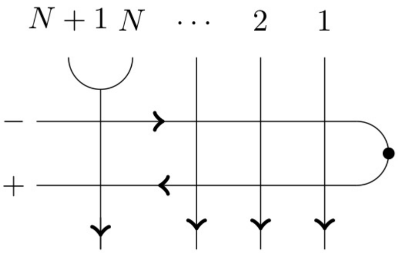
$$
\begin{aligned}
[\mathcal{B}]^{(N+1)} &= bc\,E^{+}_{-}\otimes[\mathcal{A}]^{(N)} + ab\,\mathbb{I}\otimes[\mathcal{B}]^{(N)} + bc\,E^{+}_{-}\otimes[\mathcal{D}]^{(N)} \\
&\quad +(-1)^{N+1}\left\{q_{N+1,N}\,\mathcal{B}^{(N+1)} - d^2\mathcal{B}^{(N)}\,q_{N+1,N}\right\}
\end{aligned}
$$

- Then consider

$$q_{N+1,N}\,\mathcal{B}^{(N+1)} - d^2\mathcal{B}^{(N)}\,q_{N+1,N} =$$



$-d^2$



=> $$q_{N+1,N}\,\mathcal{B}^{(N+1)} - d^2\mathcal{B}^{(N)}\,q_{N+1,N} = ab(ab-d^2)E^{++}_{-}\otimes B^{(N-1)},$$

- The inductive hypoth. then gives result.

4. Consequences of Theorem

Theorem : *We have the following commutation relations for* $N \geq 2$*:*

$$\begin{aligned}
Q^{(N)}\mathcal{A}^{(N)} - d^2\mathcal{A}^{(N-1)}Q^{(N)} &= (-1)^N c\, d\, E^+ \otimes \mathcal{B}^{(N-1)},\\
Q^{(N)}\mathcal{B}^{(N)} - d^2\mathcal{B}^{(N-1)}Q^{(N)} &= 0,\\
Q^{(N)}\mathcal{C}^{(N)} - d^2\mathcal{C}^{(N-1)}Q^{(N)} &= (-1)^N \Big(a\, c\, E^+ \otimes \mathcal{A}^{(N-1)} + c^2\, E^- \otimes \mathcal{B}^{(N-1)} + c\, d\, E^+ \otimes \mathcal{D}^{(N-1)}\Big),\\
Q^{(N)}\mathcal{D}^{(N)} - d^2\mathcal{D}^{(N-1)}Q^{(N)} &= (-1)^N b\, c\, E^+ \otimes \mathcal{B}^{(N-1)}.
\end{aligned}$$

- $$t^{(N)}(u) = \frac{\sinh(u)}{\sinh(\eta)}\mathcal{A}^{(N)}(u) - \frac{\sinh(u+2\eta)}{\sinh(\eta)}\mathcal{D}^{(N)}(u),$$

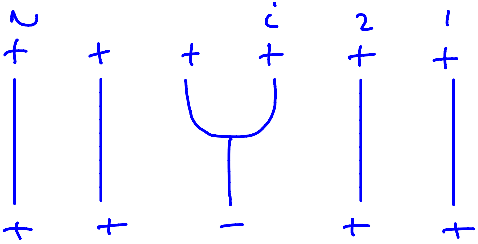
$$\Rightarrow \quad Q^{(N)}t^{(N)}(u) = d(u)^2\, t^{(N-1)}(u)\, Q^{(N)}. \quad \Rightarrow \quad H^{(N-1)}Q^{(N)} = Q^{(N)}H^{(N)}$$

- $$Q^{(N)}\mathcal{B}^{(N)}(\lambda_1)\mathcal{B}^{(N)}(\lambda_2)\cdots\mathcal{B}^{(N)}(\lambda_m)\,\Omega^{(N)} = \prod_{i=1}^{m} d(\lambda_i)^2\,\mathcal{B}^{(N-1)}(\lambda_1)\mathcal{B}^{(N-1)}(\lambda_2)\cdots\mathcal{B}^{(N-1)}(\lambda_m)Q^{(N)}\,\Omega^{(N)}.$$

$$\Omega^{(N)} = v_+ \otimes v_+ \otimes \cdots \otimes v_+$$

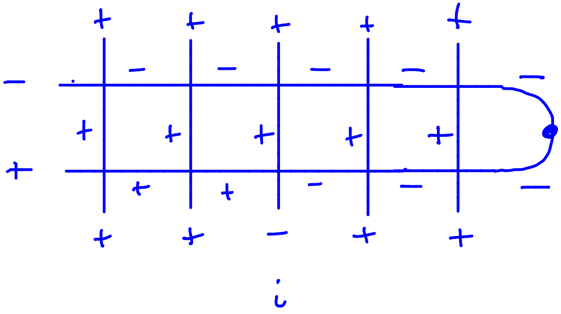
- Also note $$Q^{(N)}\,\Omega^{(N)} = \sum_{i=1}^{N-1}(-1)^{i+1}\sigma_i^-\Omega^{(N-1)} = (-1)^N\mathcal{B}^{(N-1)}(\eta)\Omega^{(N-1)}.$$

$$Q^{(N)}\Omega^{(N)} = \sum_{i=1}^{N-1}(-1)^{1+i}$$



and $R(\eta) = \begin{pmatrix} -1 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & -1 \end{pmatrix}$, $K(\eta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\Rightarrow$

$$B^{(N-1)}\Omega^{(N-1)} = \sum_{i=1}^{N-1}$$



- Hence

$$Q^{(N)}\mathcal{B}^{(N)}(\lambda_1)\mathcal{B}^{(N)}(\lambda_2)\cdots\mathcal{B}^{(N)}(\lambda_m)\,\Omega^{(N)}$$
$$=(-1)^N\prod_{i=1}^{m}d(\lambda_i)^2\,\mathcal{B}^{(N-1)}(\lambda_1)\mathcal{B}^{(N-1)}(\lambda_2)\cdots\mathcal{B}^{(N-1)}(\lambda_m)\mathcal{B}^{(N-1)}(\eta)\,\Omega^{(N-1)}.$$

i.e. takes off-shell Bethe state $\lambda_i \neq \eta$ to another

<u>NB</u> $d(\eta) = 0$

- Proposition : If $\{\lambda_1, .., \lambda_m\}$, $\lambda_i \neq \eta$ satisfy the B. eqns

$$-\sinh(2\lambda_j)\frac{\Delta_+^{(N)}(\lambda_j)}{\Delta_-^{(N)}(\lambda_j)}=\prod_{k\neq j}\frac{\sinh(\lambda_j-\lambda_k+\eta)\sinh(\lambda_j+\lambda_k+2\eta)}{\sinh(\lambda_j-\lambda_k-\eta)\sinh(\lambda_j+\lambda_k)}$$

then so do $\{\lambda_1, \ldots, \lambda_m, \eta\}$.

So on-shell → on-shell.

5. The Ground State – Some combinatorics

- A ground state $\psi^{(N)} \in V^{\otimes N}$ is SUSY singlet

$$Q^{(N)} \psi^{(N)} = 0 , \qquad Q^{(N+1)\dagger} \psi^{(N)} = 0 .$$

which means $\psi^{(N)} \in \ker(Q^{(N)})$ & $(Q^{(N+1)} \gamma^{(N+1)} , \psi^{(N)}) = 0$

$\forall \; \gamma^{(N+1)} \in V^{\otimes N+1}$

- [Hagendorf & Liénardy(16)] recently showed that

(i) $\psi^{(N)}$ is unique

(ii) $\psi^{(N)} = \chi^{(N)} - \underline{I}^{(N)}$

where $\chi^{(N)} = v_+ \otimes v_- \otimes v_+ \otimes v_- \otimes \cdots \in \ker(Q^{(N)})$

$\underline{I}^{(N)} =$ unknown state $\in \underline{I}m(Q^{(N+1)})$

i.e. $\psi^{(N)} \in [\chi^{(N)}] = \ker(Q^{(N)}) / \mathrm{Im}(Q^{(N+1)})$

- Suppose we have basis $\{\omega_i^{(N)}\}$ for $\mathrm{Im}(Q^{(N+1)})$

Then $(\omega_i^{(N)}, \psi^{(N)}) = (Q^{(N+1)}\gamma_i, \psi^{(N)}) = (\gamma_i, Q^{(N+1)\dagger}\psi^{(N)})$

$= 0$

But $\psi^{(N)} = \chi^{(N)} - \sum_j c_j \omega_j^{(N)}$

$\Rightarrow \quad (\omega_i^{(N)}, \chi^{(N)}) = \sum_j (\omega_i^{(N)}, \omega_j^{(N)}) c_j = \sum_j A_{ij} c_j$

$\Rightarrow \quad c_j$ & hence $\psi^{(N)}$.

- So far no explicit expression for $\{\omega_i\}$, but nice combinatorics !

- Note that we actually just need basis of

$\mathrm{Im}(Q^{(N+1)}$ spin 0 $\quad N$ even

$\mathrm{Im}(Q^{(N+1)})$ spin 1 $\quad N$ odd.

- let $i_N = \dim(\operatorname{Im} Q^{(N+1)})$, $k_N = \dim(\ker Q^{(N)})$

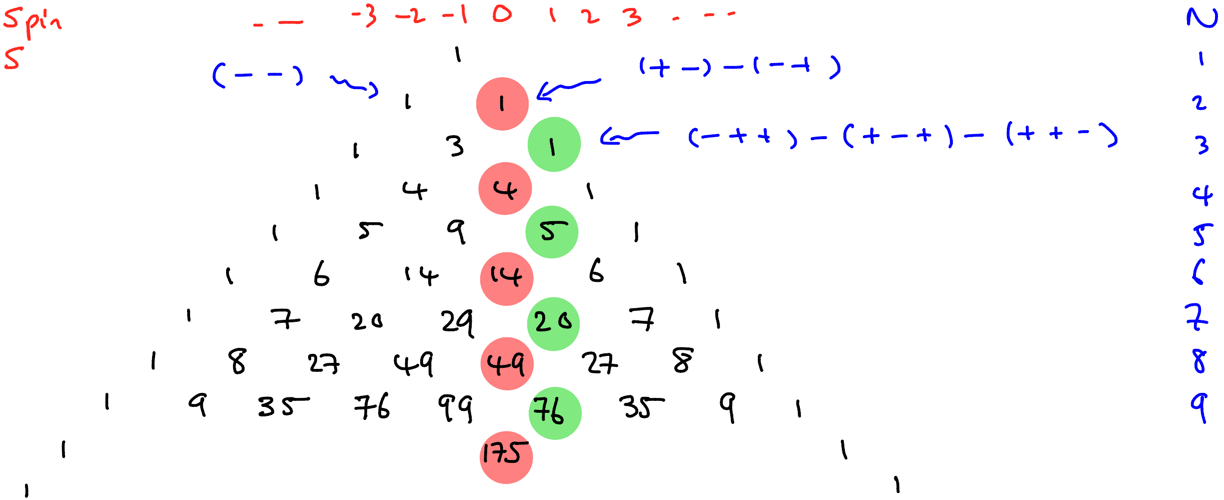
Then $i_{N-1} + k_N = 2^N$; $k_N = i_N + 1$

$\Rightarrow$ $i_{N-1} + i_N = 2^N - 1$ $\Rightarrow$ $i_{2m} = \frac{2}{3}(2^{2m} - 1)$;

$i_{2m+1} = \frac{2}{3}(2^{2m+1} - 1) + \frac{1}{3}$

or $i_n = \text{ceil}\left(\frac{2}{3}(2^n - 1)\right)$ = 1, 2, 5, 10, 21, 42, 85, 170, 341, . . .

- Can decompose $\operatorname{Im}(Q^{(N+1)}) \subset V^{\otimes N}$ with respect to spin



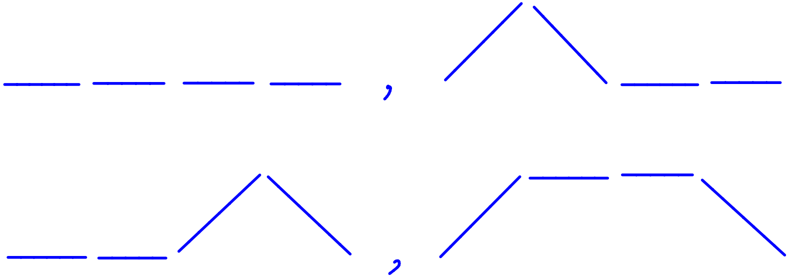
- These #s are v. combinatorial e.g.

$$d_n = \dim\left(\mathrm{Im}\left(Q^{2n+1}\right)\right)_{\text{spin } 0} =$$

$$= 1,\ 4,\ 14,\ 49,\ 175,\ \cdots \ .$$

Count Motzkin paths of length $2n$ with exactly 1 flat portion $= d_{n-1} + c(n-1)(2n-1) = \cdots$

$\underline{\underline{n=2}}$ $d_2 = 4$,



- So $\left(\mathrm{Im}\ Q^{(2n+1)}\right)_{\text{spin } 0} = \{ w_{p_n} \mid p_n \in \text{above set} \}$.

No w_{p_n} yet!

6. Summary / Comments

- We will use main result in connecting SUSY to existing algebraic picture of open chains.

- Optimistic our g.s. analysis will yield explicit formula.

- $$Q^{(N)}\mathcal{B}^{(N)}(\lambda_1)\mathcal{B}^{(N)}(\lambda_2)\cdots\mathcal{B}^{(N)}(\lambda_m)\,\Omega^{(N)}$$
  $$=(-1)^N\prod_{i=1}^{m}d(\lambda_i)^2\,\mathcal{B}^{(N-1)}(\lambda_1)\mathcal{B}^{(N-1)}(\lambda_2)\cdots\mathcal{B}^{(N-1)}(\lambda_m)\mathcal{B}^{(N-1)}(\eta)\,\Omega^{(N-1)}.$$

means we can also characterise singlet g.s. in terms of Ker $\overset{(N-1)}{B}(\eta)$ / Im $B(\eta)^{(N-1)}$. Perhaps useful.

- It would be good to approch g.s. using other tools, e.g. qKZ.