HERIOT WATT UNIVERSITY

Conserved Currents & Discrete Holomorphicity in Solvable Vertex & Face Models

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'New Trends in Integrable Models'

IIP, Natal

– Sep 2nd, 2016 –

Plan

1. Background
2. Conserved Currents & Discrete Holomorphicity
3. Conserved Currents from Quantum Groups
4. The 6-Vertex Model
5. A Use of the Construction
6. Trigonometric SOS Models ← Main example
7. Summary / Conclusions

Based on work:
Ikhlef, RW, Wheeler, Zinn-Justin (13)
Ikhlef, RW (15), (16)

1. Background

- 2D CFT understood since 80s
  - many critical lattice models linked to CFTs

- 1st rigorous results from 2000 on [see 2010 Smirnov review]

- Full results for Ising, plus v. few other cases: dimers, percolation ...

- Proofs rely on existence of a DH lattice operator: corr fns obey a discrete BVP; cont. limit exists & is unique

- Beyond Ising: Ops obeying $\frac{1}{2}$ of DH conditions constructed 'by hand' for a few models [$O(N)$ & $\mathbb{Z}(N)$]

  Cardy, Riva, Rajabpour, Ikhlef (06–);
  de Gier, Lee, Rasmussen (13);
  Alam, Batchelor (14)

- Understood how to construct in terms of Quantum Gps

  Ikhlef, RW, Wheeler, Zinn-Justin (13)

- $\frac{1}{2}$DH still enough to prove some aspects of scaling limit

  Duminil-Copin & Smirnov (10)

  Also useful for identifying PCFT related to massive lattice models

2. Conserved Currents & DH

- Cauchy-Riemann relns: For $J(x,t) = J^x(x,t) + iJ^t(x,t)$

CR relns are

$$\partial_t J^t - \partial_x J^x = 0 \qquad (1)$$

$$\partial_x J^t + \partial_t J^x = 0 \qquad (2)$$

or with $z = x + it$, $J = J^x + iJ^t$

$$\partial_{\bar{z}} J - \partial_z \bar{J} = 0 \qquad (1)$$

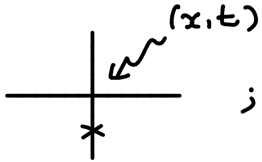
$$\partial_{\bar{z}} J + \partial_z \bar{J} = 0 \qquad (2)$$

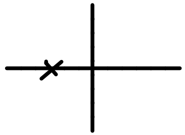
(1) is current conservation

(2) isn't

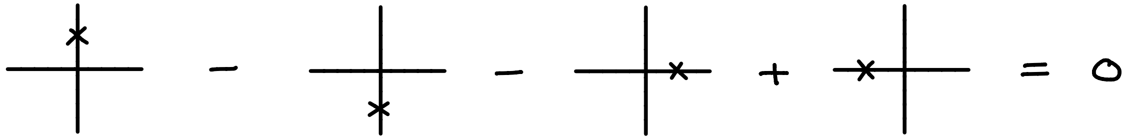
- Discretization

- Consider J defined at midpoint of edges of lattice:

$J^t_{(x,t)} \sim$ 

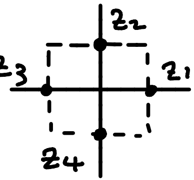
; $J^x_{(x,t)} \sim$ 

- A possible discretization of $\partial_t J^t - \partial_x J^x = 0$ (1) is



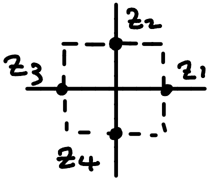
$$J^t_{(x,t+1)} - J^t_{(x,t)} - J^x_{(x+1,t)} + J^x_{(x,t)} = 0 \quad (D1)$$

Defining $J(x,t) = \begin{cases} J^x(x,t) & ; \ (x,t) \text{ on horiz. edge} \\ i J^t(x,t) & ; \quad \text{"} \quad \text{"} \ \text{vert.} \ \text{"} \end{cases}$

(D1) is $\boxed{\sum_i \delta z_i \, J(z_i, \bar{z}_i) = 0}$ around 

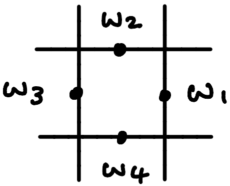
$\delta z_1 = i$, $\delta z_2 = -1$, $\delta z_3 = -i$, $\delta z_4 = 1$

- $\sum_i \delta z_i \, J(z_i, \bar{z}_i) = 0$ (D1) is discrete analog of $\oint J \, dz = 0$ around



- Similar disc. analog of $\partial_x J^t + \partial_t J^x = 0$ (D2) is

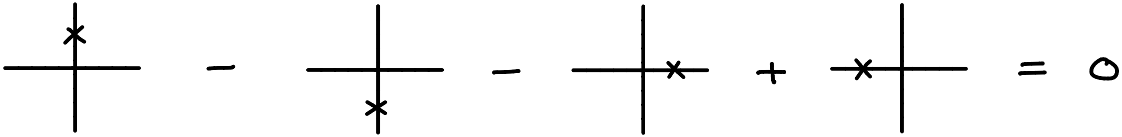
$\sum_i \delta w_i \, J(w_i, \bar{w}_i) = 0$ (D2) around



- (D1) plus (D2) are DH
- (D1) alone is current conservation

3. Conserved Currents from Quantum Groups

- Operators obeying



$$j(x,t+1) - j(x,t) - j(x+1,t) + j(x,t) = 0 \quad (D1)$$

come directly from Q. groups [Bernard, Felder (91)]

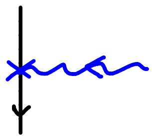
- Consider $U = U_q(\hat{sl}_2)$, gen by $e_i, f_i, t_i^{\pm 1}$ $\quad i \in \{0,1\}$

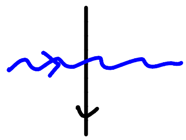
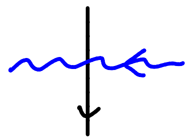
  $\Delta : U \to U \otimes U$ chosen to be

$$\Delta(f_i) = f_i \otimes t_i^{-1} + \mathbb{1} \otimes f_i \quad , \quad \ldots$$

$$\Delta^{(N)}(f_i) = \sum_{j=1}^{N} \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underset{j}{f_i} \otimes t_i^{-1} \otimes \cdots \otimes t_i^{-1}$$

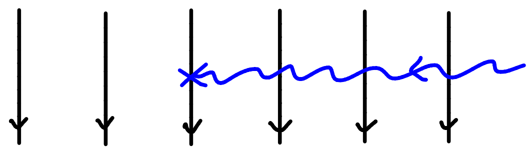
- Pictures: Denote repn V by 

f_i repn by 

t_i by  , t_i^{-1} by 

- Then

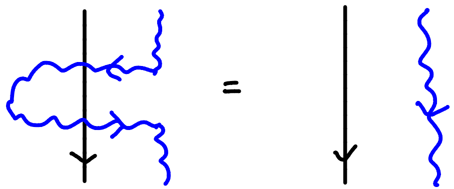
$$\Delta^{(N)}(f_i) = \sum_{j=1}^{N} \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underset{j}{f_i} \otimes t_i^{-1} \otimes \cdots \otimes t_i^{-1}$$

$\sim \quad \sum_{j=1}^{N}$ 

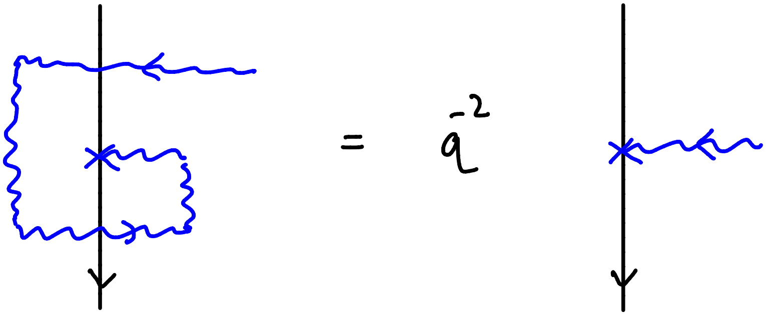
j

- Defining relns have nice form : e.g.

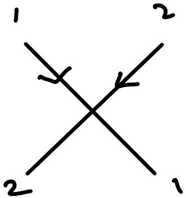
$t_i \, t_i^{-1} = \mathbb{1}$ $\sim$



$t_i f_i t_i^{-1} = q^{-2} f_i$ $\sim$

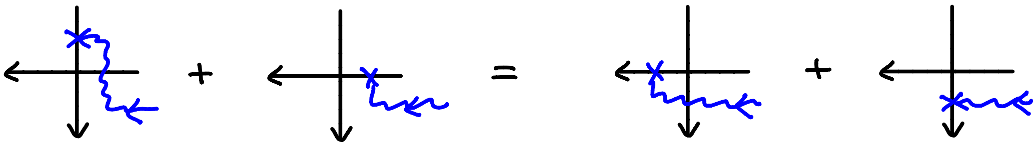


- Repn R-matrix $\check{R} : V_1 \otimes V_2 \to V_2 \otimes V_1$ by

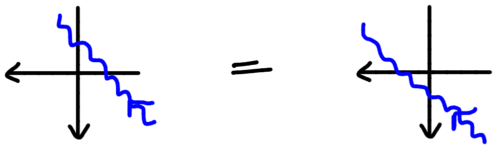
$\check{R} \sim$ 

- R-matrix satisfies $\check{R} \circ \Delta(x) = \Delta(x) \circ \check{R}$

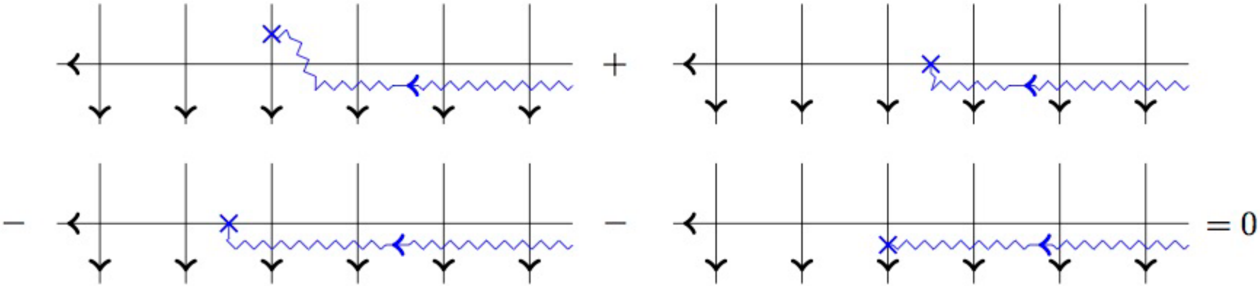
– For $\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i$



– For $\Delta(t_i^{-1}) = t_i^{-1} \otimes t_i^{-1}$



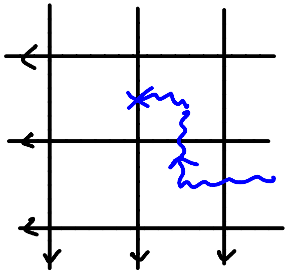
- Hence we get



- Denoting f_i insertion at (x,t) by $j_i(x,t)$, this is

$$j_i(x,t+1) - j_i(x,t) - j_i(x+1,t) + j_i(x,t) = 0 \quad (D1)$$

- Can view as reln between exp. values in 2D lattice model



4. Example + refinement to different embedding angle – the 6 Vertex model

- Algebra is $U_q(\hat{sl}_2)$ gen by $e_\ell, f_\ell, t_i^{\pm 1}$, $\ell = 0, 1$

  Interested in 2D repn V_λ

$$f_1 \sim \begin{pmatrix} 0 & 0 \\ e^{-\lambda} & 0 \end{pmatrix}, \quad f_0 \sim \begin{pmatrix} 0 & e^{-\lambda} \\ 0 & 0 \end{pmatrix}, \quad t_1 \sim e^{\eta \sigma_z}, \quad t_0 = e^{-\eta \sigma_z}$$

$$e_1 \sim \begin{pmatrix} 0 & e^{\lambda} \\ 0 & 0 \end{pmatrix}, \quad e_0 \sim \begin{pmatrix} 0 & 0 \\ e^{\lambda} & 0 \end{pmatrix}$$

$(q = e^{\eta})$

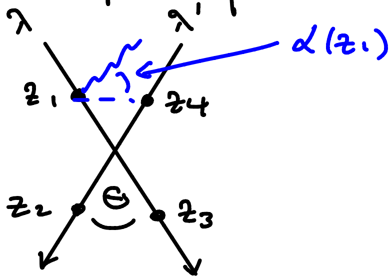
$$\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i$$

sinh

- $$R(\lambda) = \begin{pmatrix} \mathrm{sh}(\lambda+\eta) & & & \\ & \mathrm{sh}(\lambda) & \mathrm{sh}(\eta) & \\ & \mathrm{sh}(\eta) & \mathrm{sh}(\lambda) & \\ & & & \mathrm{sh}(\lambda+\eta) \end{pmatrix}$$

- embed into complex plane thus



$$\Theta = \alpha(z_1) - \alpha(z_2)$$
$$= \frac{\pi(\lambda' - \lambda)}{\eta}$$

fixed by crossing

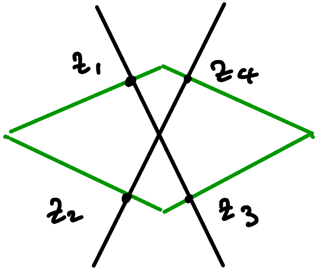
$$R(\lambda)^{\varepsilon_1 \varepsilon_2}_{\varepsilon_1' \varepsilon_2'} = R(-\lambda-\eta)^{\varepsilon_2 \, -\varepsilon_1'}_{\varepsilon_2' \, -\varepsilon_1}$$

fe current

Then $\phi_\ell(z,\bar{z}) = e^{-i\alpha(z)} \cdot j_\ell(z,\bar{z})$

satisfies $\displaystyle\sum_{j=1}^{4} \delta_{z_j} \phi_\ell(z,\bar{z}) = 0$ (DH1)

around



spec. param of line at z

Can write as $\phi_\ell(z,\bar{z}) = e^{-is\alpha(z)} e^{\lambda(z)} \cdot j_\ell(z,\bar{z})$

$s = \left(1 + \frac{i\eta}{\pi}\right)$ 'spin' with.

$\phi_0 \sim e^{-is\alpha(z)} \sigma^+ \otimes q^{\sigma_z} \otimes q^{\sigma_z} \otimes \ldots$

$\phi_1 \sim e^{-is\alpha(z)} \sigma^- \otimes q^{-\sigma_z} \otimes q^{-\sigma_z} \otimes \ldots$

- Procedure carried out for

- Dense loop models — Using 6V-loop/model connection
- Dilute loop models — Using $A_2^{(2)}$ vertex / dilute loop model connection
- Chiral Potts model — Using cyclic repns of $U_q(\hat{sl_2})$ at $q = e^{i\pi/N}$
- Trig. SOS models — Using vertex / face corresp.

- In all cases, we produce parafermionic, fractional spin ops in model.

  New in some cases.

5. A Use of the Construction

- Massless loop & $\mathbb{Z}_N$ (CP) case : PFs coincide with existing constructions

- Massive case : Analysed most for CP case

(D1) $\sum_i \delta_{z_i} J(z_i, \bar{z}_i) = 0$ is discrete $\partial_{\bar{z}} J = 0$

Expanding $J = \underbrace{J_0}_{\text{massless limit}} + \overset{\text{corr.}}{J_1} + \cdots$, get

$\partial_{\bar{z}} J_0 \sim \sum_i \lambda_i \chi_i$ ← non-local fract spin

- Standard CFT arg. $\Rightarrow$

If $S = S_{CFT} + \sum_i \lambda_i \int \phi_i(z,\bar{z})\, d^2r$, then

$\partial_{\bar{z}} \bar{J}_0 \sim \sum_i \lambda_i \chi_i$ where in CFT

Chiral field in CFT

$$J_0(z)\, \phi_i(w,\bar{w}) \sim \cdots + \frac{\chi_i(w,\bar{w})}{z-w} + \cdots$$

- By comparing we have

i) confirmed Cardy's (93) prediction of PCFT for CP

$$\left(\phi_1 = \varepsilon = \left(\tfrac{2}{N+2}, \tfrac{2}{N+2}\right),\ \phi_2 = W_{-1}\varepsilon,\ \phi_3 = \bar{W}_{-1}\varepsilon \right)$$

ii) Found λ_i in terms of CP params.

6. Trigonometric SOS Models

- Vertex/Face corres: 8-V $\longrightarrow$ elliptic SOS
  [Baxter 73]

  Taking nome $\to 0$ gives 6-V $\to$ trig. SOS

- Start from 6-Vertex

$$R(\lambda) = \begin{pmatrix} \mathrm{sh}(\lambda+\eta) & & & \\ & \mathrm{sh}(\lambda) & \mathrm{sh}(\eta) & \\ & \mathrm{sh}(\eta) & \mathrm{sh}(\lambda) & \\ & & & \mathrm{sh}(\lambda+\eta) \end{pmatrix}$$

- Introduce vector-valued fn $\psi(a,b\,|\,\lambda)$
  $a, b \in \mathbb{Z}$, $|a-b| = 1$

$$\psi(a, a\pm 1|\lambda) = \begin{pmatrix} \exp\left(-\dfrac{\lambda \pm a\eta}{2}\right) \\ \exp\left(+\dfrac{\lambda \mp a\eta}{2}\right) \end{pmatrix}$$

- Then

$$R(\lambda_1-\lambda_2)\left[\psi(a,b|\lambda_1) \otimes \psi(b,c|\lambda_2)\right]$$

$$= \sum_d \left[\psi(d,c|\lambda_1) \otimes \psi(a,d|\lambda_2\right]\, W\left(\begin{matrix} a & b \\ d & c \end{matrix} \,\middle|\, \lambda_1-\lambda_2\right)$$

SOS weight

- $$W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{matrix}\middle|\, \lambda\right) = \operatorname{sh}(\lambda+\eta)$$

$$W\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a \end{matrix}\middle|\, \lambda\right) = \frac{\operatorname{sh}(\lambda)\,\operatorname{sh}((a\pm 1)\eta)}{\operatorname{sh}(a\eta)}$$

$$W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\middle|\, \lambda\right) = \frac{\operatorname{sh}(\eta)\,\operatorname{sh}(a\eta\mp\lambda)}{\operatorname{sh}(a\eta)}$$

- or $$W\left(\begin{matrix} a & a+\varepsilon_1 \\ a+\varepsilon_2' & a+\varepsilon_1+\varepsilon_2 \end{matrix}\middle|\, \lambda\right) = \underbrace{R(\lambda;a)_{\varepsilon_1'\varepsilon_2'}^{\varepsilon_1\varepsilon_2}}_{\text{dynamical R-matrix.}}$$

with $\varepsilon_1+\varepsilon_2 = \varepsilon_1'+\varepsilon_2'$

- W obeys face / dynamical YBE $\quad (\lambda_{ij} = \lambda_i - \lambda_j)$

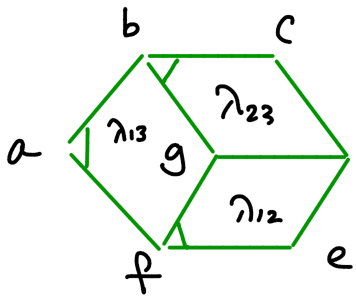
$$\sum_g W\left(\begin{matrix} f & g \\ e & d \end{matrix}\middle|\,\lambda_{12}\right) W\left(\begin{matrix} a & b \\ f & g \end{matrix}\middle|\,\lambda_{13}\right) W\left(\begin{matrix} b & c \\ g & d \end{matrix}\middle|\,\lambda_{23}\right)$$

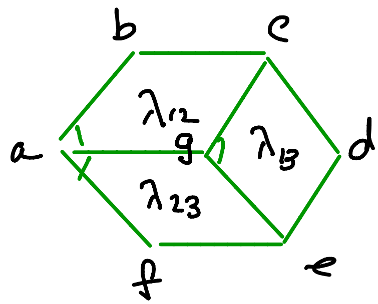
$$= \sum_g W\left(\begin{matrix} a & g \\ f & e \end{matrix}\middle|\,\lambda_{23}\right) W\left(\begin{matrix} g & c \\ e & d \end{matrix}\middle|\,\lambda_{13}\right) W\left(\begin{matrix} a & b \\ g & c \end{matrix}\middle|\,\lambda_{12}\right)$$

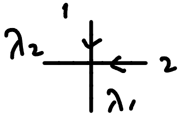
or

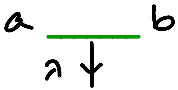
$$R_{12}(\lambda_{12}; a+\sigma_3^z)\, R_{13}(\lambda_{13}; a)\, R_{23}(\lambda_{23}; a+\sigma_1^z)$$

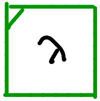
$$= R_{23}(\lambda_{23}; a)\, R_{13}(\lambda_{13}; a+\sigma_2^z)\, R_{12}(\lambda_{12}; a)$$

$\sum_g$ 

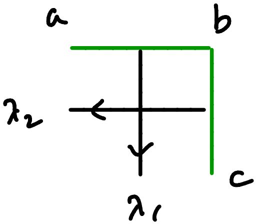
$=$ $\sum_g$ 

- With $R(\lambda_1 - \lambda_2) \sim$ 

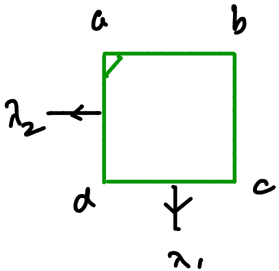
  $\psi(a,b|\lambda) \sim$ 

  $W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \middle| \lambda \right) \sim$ 

- $$R(\lambda_1 - \lambda_2)\left[\psi(a,b|\lambda_1) \otimes \psi(b,c|\lambda_2)\right]$$
  $$= \sum_d \left[\psi(d,c|\lambda_1) \otimes \psi(a,d|\lambda_2)\right] W\left(\begin{matrix} a & b \\ d & c \end{matrix} \middle| \lambda_1 - \lambda_2\right)$$ is



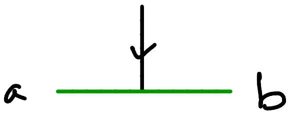
$= \sum_d$



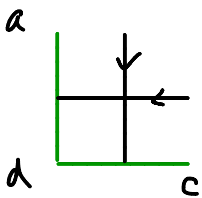
- Also useful to introduce $\psi^*(a,b|\lambda)$

$$\psi^*(a, a\pm 1|\lambda) = \frac{\pm 1}{2\,\mathrm{sh}(a\eta)}\left[\exp\left(\frac{\lambda \pm a\eta}{2}\right), -\exp\left(\frac{-\lambda \mp a\eta}{2}\right)\right]$$

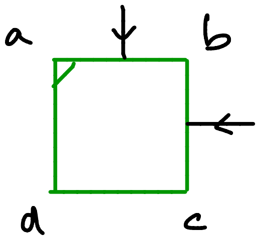
$\psi^*(a,b|\lambda) \sim$



obeying

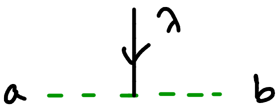


$$= \sum_b$$



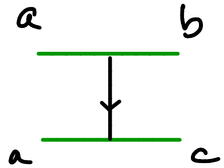
- Finally, modify $\psi'(a,b|\lambda) = \dfrac{\mathrm{sh}(a\eta)}{\mathrm{sh}(b\eta)}\,\psi^*(a,b|\lambda)\,e^{\eta\sigma_z}$

$\sim$



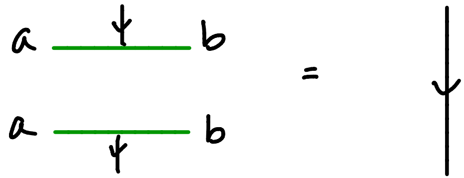
- 4 inversion relns hold

$$\psi^*(a,c|\lambda)\,\psi(a,b|\lambda) = \delta_{b,c}$$

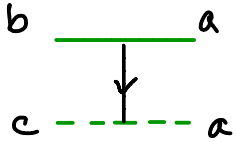


$= \delta_{b,c}$

$$\sum_b \psi(a,b|\lambda)\,\psi^*(a,b|\lambda) = \mathbb{I}$$

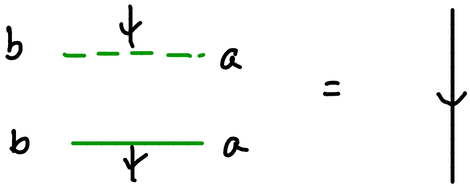


$$\psi'(c,a|\lambda)\,\psi(b,a|\lambda) = \delta_{b,c}$$



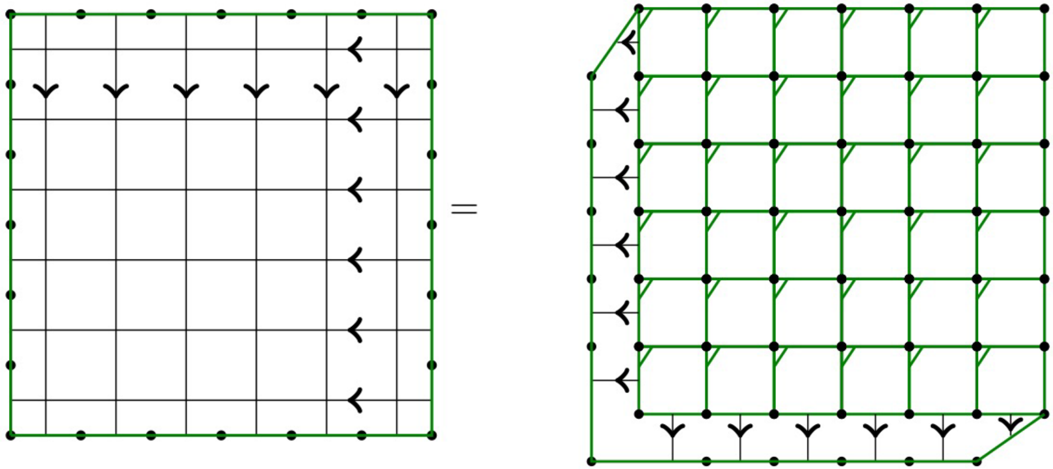
$= \delta_{b,c}$

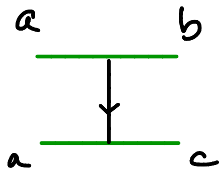
$$\sum_b \psi(b,a|\lambda)\,\psi'(b,a|\lambda) = \mathbb{I}$$



- With these relns, we can map part. fn & corr fns from vertex → face.

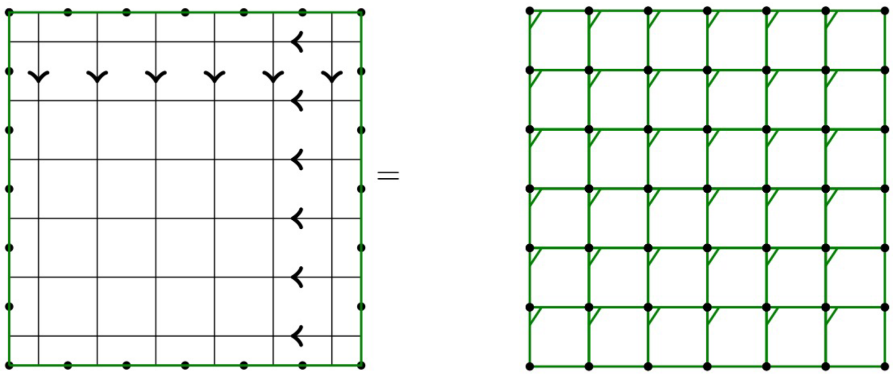
- e.g. dressed partition fn



- Now we 

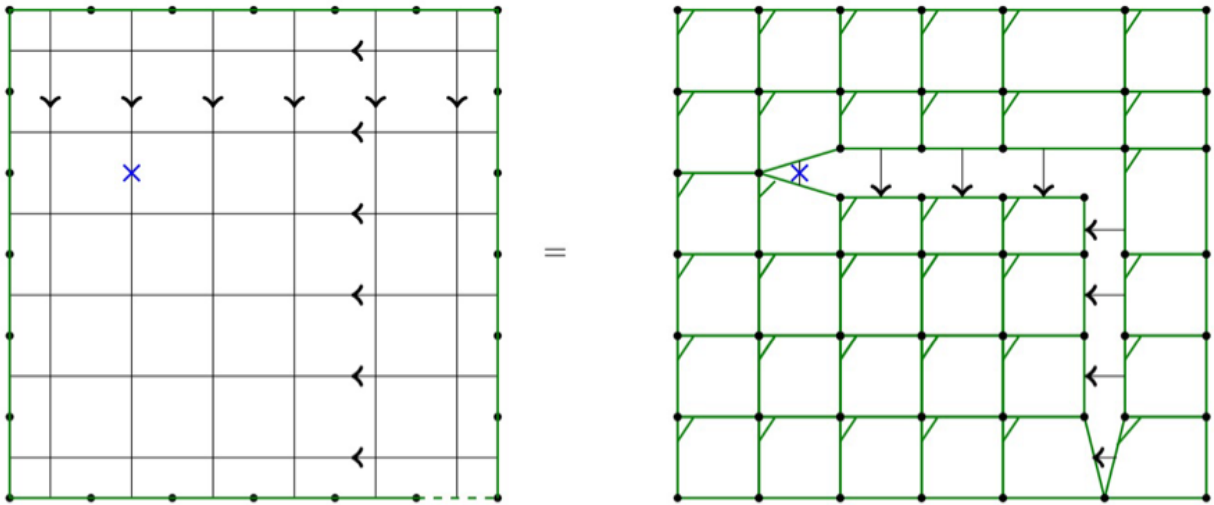
$= \delta_{b,c}$

- Hence



- This also works for corr fns of local operators

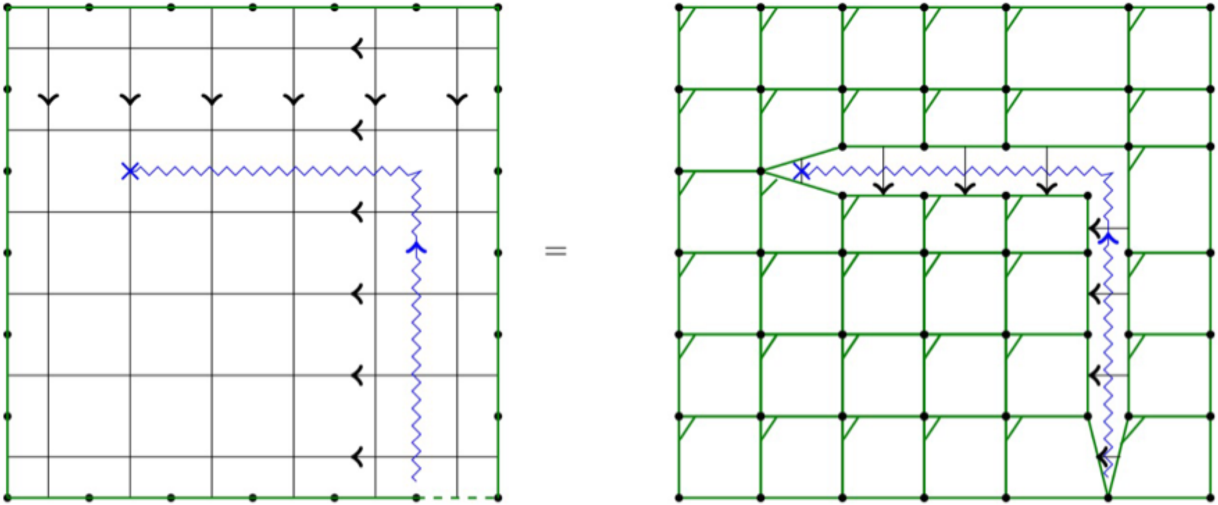
  e.g.



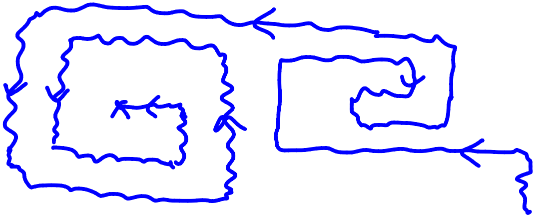
NB local ops in 6-V model become non-local ones in SOS model.

- Used by [Lashkevich / Pugai 1982] to get corr fns of 8-V model using vertex operator approach.

- For 6V non-local ops with simple tail, work the same



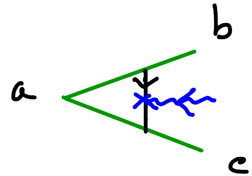
- But what about complicated tail configs?



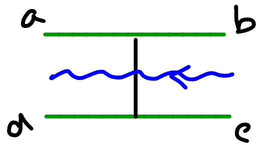
- It turns out that we <u>can</u> force V-IRF corresp. through all such configs.
- Need all 4 inversion relns

- End up with 'plumbing' rules for building pure SOS currents from dressed f_i, $t_i^{\pm 1}$

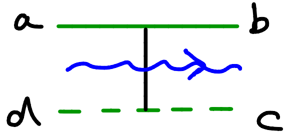
$$F_i\left(a\ {}^{b}_{c} \middle| \lambda\right) = \psi^*(a,c|\lambda)\, f_i\, \psi(a,b|\lambda) =$$



$$T_i^-\left({}^{a}_{d}\ {}^{b}_{c} \middle| \lambda\right) = \psi^*(d,c|\lambda)\, t_i^{-1}\, \psi(a,b|\lambda) =$$



$$T_i^+\left({}^{a}_{d}\ {}^{b}_{c} \middle| \lambda\right) = \psi'(d,c|\lambda)\, t_i\, \psi(a,b|\lambda) =$$



- Dressed Chevalley gen obey relns analogous to $f_\ell,\ t_\ell^{\pm 1}$

  - $t_\ell f_\ell t_\ell^{-1} = q^{-2} f_\ell \quad \to$

$$\sum_{d,e} \frac{sh(d\eta)}{sh(a\eta)}\, T_\ell^{+}\left(\begin{smallmatrix} d & e \\ a & c \end{smallmatrix}\middle|\lambda\right) F_\ell\left(d\ \begin{smallmatrix} c \\ e \end{smallmatrix}\middle|\lambda\right) T_\ell^{-}\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\middle|\lambda\right)$$

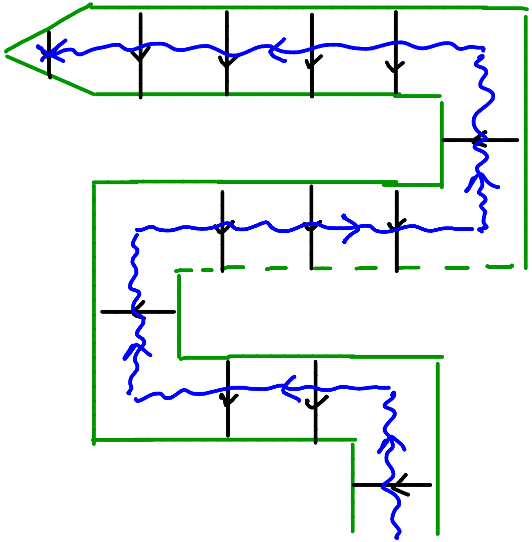
$$= q^{-4\ell} F_\ell\left(a\ \begin{smallmatrix} b \\ c \end{smallmatrix}\middle|\lambda\right)$$

- $t_\ell^{+1} t_\ell^{-1} = 1 \quad \to$

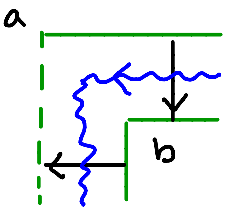
$$\sum_{e} T_\ell^{+}\left(\begin{smallmatrix} d & e \\ c & a \end{smallmatrix}\middle|\lambda\right) T_\ell^{-}\left(\begin{smallmatrix} b & a \\ d & e \end{smallmatrix}\middle|\lambda\right) = \delta_{b,c}$$

- Zip above together to get currents $J_i(z,\bar{z})$

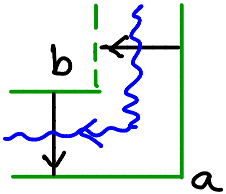
e.g.



- For $W \to S$ & $S \to W$ corners, need to add in addit. factors into $J_i(z,\bar{z})$



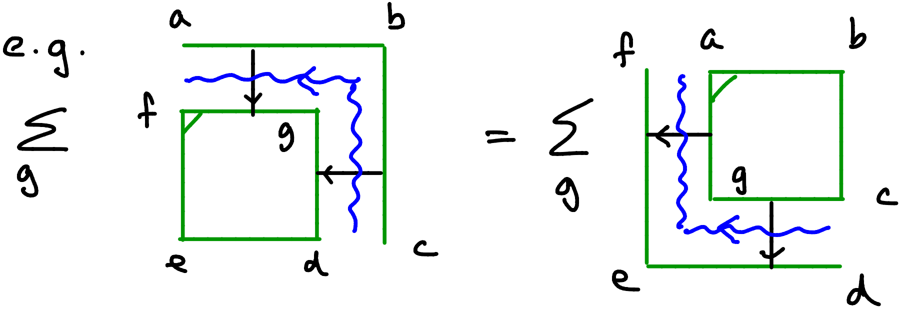
$\times \dfrac{sh(b\eta)}{sh(a\eta)}$

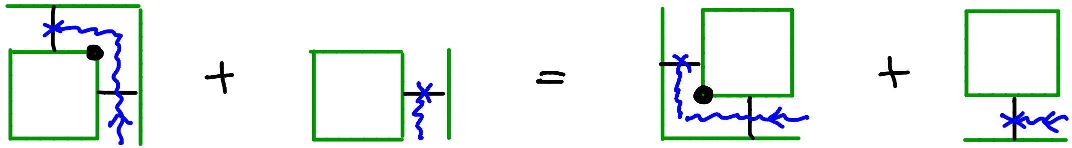


$\times \dfrac{sh(b\eta)}{sh(a\eta)}$

- Rules for moving tail through SOS weight + 4 term reln are inherited from vertex model

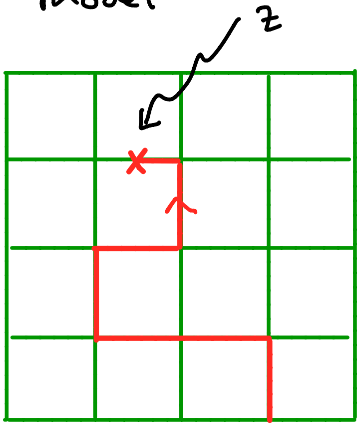
e.g.





• = summed over

- We can view $J_\ell(z,\bar z)$ as living on seams of pure SOS model

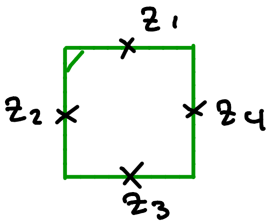


With this defn, under VFC

$$j_\ell(z,\bar z) \sim e^{2N(1-2\ell)\eta} J_\ell(z,\bar z)$$

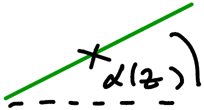
N = winding # of tail

- Around



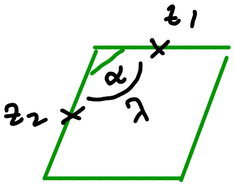
$$J(z_1) - J(z_2) - J(z_3) + J(z_4) = 0$$

- Let



$\lambda(z) = \lambda$ spectral variable

with



$$\lambda = \lambda(z_1) - \lambda(z_2) = \frac{\eta\,(\alpha(z_1) - \alpha(z_2))}{\pi} = \frac{\eta\alpha}{\pi}$$

$\uparrow$ fixed by crossing symm

- led to define

$$\Phi_i(z,\bar z) = e^{-iS_i\alpha(z)}\left[e^{+D_i\lambda(z)} J_i(z,\bar z) \right]$$

$$D_i = i(1 - S_i)\pi/\eta$$

where S_i = 'spin'

$$S_0 = 1, \qquad S_1 = 1 + \frac{2i\eta}{\pi}$$

with
$$\sum_{j=1}^{4} \delta z_j\, \Phi_i(z_j, \bar z_j) = 0 \qquad (DH1)$$

7. Summary / Conclusions

- Quantum gp currents lead to parafermionic ops obeying DH1
- VF corr allows us to obtain parafermionic fields in SOS models
- CFT / PCFT interpretation can be constructed (still in prog for SOS)
- DH2 still missing, except in Ising case (ϕ_0 & ϕ_1 coincide giving DH1 & DH1′)