Question 1 (20 Marks)
(a) Suppose the function $w(t)$ is defined in a neighbourhood of $t = t_0$. Define what is meant by:

(i) $f(t) = \mathcal{O}(w(t))$ as $t \to t_0$;
(ii) $g(t) = o(w(t))$ as $t \to t_0$;
(iii) $h(t) \sim w(t)$ as $t \to t_0$.

(b) Prove that if $x = \mathcal{O}(t^2)$ and $y = o(t^6)$ as $t \to 0^+$, then $xy^{\frac{1}{2}} = o(t^5)$ as $t \to 0^+$.

(c) Determine the range of values of $p$ for which

$$\sin^2(x) = \mathcal{O}(x^p) \quad \text{as} \quad x \to 0.$$

(d) Explain what is precisely meant by the statements:

(i) $\{\varphi_n(x)\}$ is an asymptotic sequence of functions as $x \to x_0$;
(ii) $f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x)$ as $x \to x_0$.

(e) Prove that every Taylor series (with a positive radius of convergence about $x_0$) is also an asymptotic power series as $x \to x_0$.

Question 2 (30 Marks)
(a) Use an appropriate power series expansion to find an asymptotic approximation as $\epsilon \to 0^+$, correct to $\mathcal{O}(\epsilon^3)$, for the two roots close to $+1$ of

$$\epsilon x^3 - x^2 + 2x - 1 = 0.$$

Then by using a suitable rescaling, find the first three terms of an asymptotic expansion as $\epsilon \to 0^+$ of the singular root.

(b) For $\epsilon > 0$ sufficiently small the transcendental equation

$$\epsilon e^{y^2} = y,$$

has one small and one large real root.

(i) Use the expansion method to find an asymptotic approximation correct to $\mathcal{O}(\epsilon^7)$ for the small real solution as $\epsilon \to 0^+$ (hint: you only need to expand in odd powers of $\epsilon$).

(ii) Use an iterative method to show that an asymptotic approximation for the large (and singular) real solution as $\epsilon \to 0^+$ is

$$y(\epsilon) \sim + \sqrt{\ln \left( \frac{1}{\epsilon} \right)} + \frac{1}{4} \ln \ln \left( \frac{1}{\epsilon} \right).$$
Question 3 (30 Marks)
(a) The Exponential integral function Ei(x) is defined for all \( x > 0 \) by
\[
Ei(x) \equiv \int_x^\infty \frac{e^{-t}}{t} \, dt.
\]
Prove that
\[
Ei(x) \sim e^{-x} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{x^n} \quad \text{as} \quad x \to +\infty.
\]
(b) The modified Bessel function \( K_\nu(x) \) is defined for all real \( \nu \) by
\[
K_\nu(x) \equiv \int_0^\infty e^{-x \cosh(t)} \cosh(\nu t) \, dt.
\]
Use Laplace’s method to show that
\[
K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} \cdot e^{-x} \quad \text{as} \quad x \to +\infty.
\]
(c) Use the method of stationary phase to show that
\[
\int_{-\pi/2}^{\pi/2} \sin(x \cos(t)) \, dt \sim \sqrt{\frac{2\pi}{x}} \sin \left( x - \frac{\pi}{4} \right) \quad \text{as} \quad x \to +\infty.
\]

Question 4 (20 Marks)
According to Darcy’s law the velocity of a gas flowing through a porous medium is given by
\[ v = -k \nabla p, \]
where \( p \) is the pressure of the gas and \( k \) is some positive constant.

(a) Use conservation of mass to show that
\[
\frac{\partial \rho}{\partial t} = k \nabla \cdot (\rho \nabla p).
\]
(b) (i) Taking pressure to be proportional to a power of density, show that
\[
\frac{\partial \rho}{\partial t} = K \nabla \cdot (\rho^n \nabla \rho),
\]
where \( n \) is the power appearing in the pressure–density relation, and \( K \) is related to \( n, k \) and the constant of proportionality appearing in the pressure density–relation.
(ii) By scaling the density appropriately show that the equation in part (i) reduces to the porous-medium equation
\[
\frac{\partial u}{\partial t} = \nabla \cdot (u^n \nabla u).
Formulae

Power series

\[ e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \text{ for all } x \]

\[ \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \text{ for all } x \]

\[ \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \text{ for all } x \]

\[ \sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots \text{ for all } x \]

\[ \cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \text{ for all } x \]

\[ (a + x)^k = a^k + ka^{k-1}s + \frac{k(k-1)}{2!}a^{k-2}s^2 + \cdots \text{ for } |x| < a \]

\[ \log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \text{ for } |x| < 1 \]

Perturbation expansions

\[ x(\epsilon) = x_0 + x_1 \epsilon + x_2 \epsilon^2 + x_3 \epsilon^3 + x_4 \epsilon^4 + O(\epsilon^5) \]

\[ (x(\epsilon))^2 = x_0^2 + 2x_0x_1 \epsilon + (x_1^2 + 2x_0x_2)\epsilon^2 + (2x_1x_2 + 2x_0x_3)\epsilon^3 + (x_2^2 + 2x_1x_3 + 2x_0x_4)\epsilon^4 + O(\epsilon^5) \]

\[ (x(\epsilon))^3 = x_0^3 + 3x_0^2x_1 \epsilon + 3(x_0x_1^2 + x_0^2x_2)\epsilon^2 + (x_1^3 + 6x_0x_1x_2 + 3x_0^2x_3)\epsilon^3 + 3(x_1^2x_2 + 2x_0x_1x_3 + x_0x_2^2 + x_0^2x_4)\epsilon^4 + O(\epsilon^5) \]

\[ (x(\epsilon))^4 = x_0^4 + 4x_0^3x_1 \epsilon + 2(3x_0^2x_1^2 + 2x_0^3x_2)\epsilon^2 + 4(x_0x_1^3 + 3x_0^2x_1x_2 + x_0^3x_3)\epsilon^3 + (x_1^4 + 12x_0x_1^2x_2 + 6x_0^2x_2^2 + 12x_0x_1x_3 + 4x_0^3x_4)\epsilon^4 + O(\epsilon^5) \]
Definite integrals

\[ \int_0^\infty s^{n-1} e^{-s} \, ds = n! \quad \text{for } n = 0, 1, 2, 3, \ldots \]

\[ \int_0^\infty s^{\alpha-1} e^{-s} \, ds = \Gamma(\alpha) \quad \text{for } \alpha > 0 \]

\[ \int_{-\infty}^{\infty} e^{-s^2} \, ds = \sqrt{\pi} \]

\[ \int_{-\infty}^{\infty} s^2 e^{-s^2} \, ds = \frac{\sqrt{\pi}}{2} \]

\[ \int_{-\infty}^{\infty} e^{\pm is^2} \, ds = \sqrt{\pi} e^{\pm i\pi/4} \]