Asymptotic Analysis F13YQ1: Final Exam 2002 Model Answers

■ Question 1

(a)

(i) \( f(t) = O(w(t)) \) as \( t \to t_0 \) \( \Rightarrow \exists A > 0, B > 0 \) s.t. \( |f(t)| < B|w(t)| \) for \( |t - t_0| < A \).

(ii) \( g(t) = o(w(t)) \) as \( t \to t_0 \) \( \Rightarrow \frac{g(t)}{w(t)} \to 0 \) as \( t \to t_0 \).

(iii) \( g(t) - w(t) \) as \( t \to t_0 \) \( \Rightarrow \frac{g(t)}{w(t)} \to 1 \) as \( t \to t_0 \).

(b) Since \( x = O[t]^3 \) and \( y = o[t]^4 \) as \( t \to \infty \), we know that

\[ \exists A > 0, B > 0 \text{ s.t. } |x(t)| < B|t^3| \text{ for } t > A, \]

\[ \frac{y(t)}{t^4} \to 0 \text{ as } t \to \infty. \]

\[ \Rightarrow \left| \frac{x^2 y}{t^{10}} \right| = \left| \frac{x^2}{t^6} \frac{y}{t^4} \right| < B^2 \frac{y}{t^4} \text{ for } t > A. \]

The RHS \( \to 0 \) as \( t \to \infty \), giving the required result.

(c) Using the two Taylor series for \( \cos(x) \) and \( \cosh(x) \):

\[ \cos[x] = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O[x]^5 \] \( \& \) \[ \cosh[x] = 1 + \frac{x^2}{2} + \frac{x^4}{24} + O[x]^5 \]

\[ \Rightarrow \frac{\cosh[x] - \cos[x]}{x^2} \to 1 \text{ as } x \to 0. \]

i.e. \( \exists A > 0 \text{ s.t. } \forall |x| < A, \frac{1}{2} < \left| \frac{\cosh[x] - \cos[x]}{x^2} \right| < 2. \]

Consider

\[ \left| \frac{\cosh[x] - \cos[x]}{x^p} \right| = \left| \cosh[x] - \cos[x] \right| \cdot \frac{1}{x^p - 2}. \]

\( p < 2 \Rightarrow \left| \frac{\cosh[x] - \cos[x]}{x^p} \right| < 2A^{2-p} \)
\[ p > 2 \implies \left| \frac{\cosh[x] - \cos[x]}{x^p} \right| > \frac{1}{2} \frac{1}{x^{p-2}} \]

where the lower bound on the right is unbounded as \( x \to 0 \). Hence the statement is true for \( p < 2 \).

(d) Graphing \( e^{-x} \) and \( \epsilon/x \):

\[
\text{In}[76]:= \text{Plot}[[x^{-x}, (\epsilon/x) /. \{\epsilon \to 0.2\}], \{x, 0.1, 5\}] \]

There are two roots, we're interested in the larger one. Rearranging the equation into the form

\[
e^{-x} = \frac{\epsilon}{x} \implies -x = \ln\left(\frac{\epsilon}{x}\right) \implies x = \ln[1/\epsilon] + \ln[x] .
\]

This suggests the iterative scheme

\[
x_{n+1} = \ln[1/\epsilon] + \ln[x_n] .
\]

Since \( x \gg \ln[x] \) for \( x \) large, a good initial guess is

\[
x_0 = \ln[1/\epsilon] .
\]

\[
\Rightarrow x_1 = \ln[1/\epsilon] + \ln[\ln[1/\epsilon]] = L_1 + L_2
\]

where we set

\[
L_1 = \ln[1/\epsilon] \text{ and } L_2 = \ln[\ln[1/\epsilon]] .
\]

Iterating again,
\[ x_2 = L_1 + \ln[L_1 + L_2] = L_1 + \ln\left[ L_1 \left( 1 + \frac{L_2}{L_1} \right) \right] \]

\[ \Rightarrow x_2 = L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2 L_1^2} + \ldots \]

i.e. \[ x_2 = \ln[1/e] + \ln[\ln[1/e]] + \frac{\ln[\ln[1/e]]}{\ln[1/e]} + \ldots . \]

**Question 2**

(a) First we solve the given quadratic equation with \( e=0 \):

\[
\begin{align*}
P_2[x_\infty, e_\infty] &:= (1 - e) x^2 + 2 (1 + e) x + 1; \\
\text{Solve}[P_2[x, e] = 0 / . e \rightarrow 0, x] & \Rightarrow \{x \rightarrow 1\}, \{x \rightarrow 1\}
\end{align*}
\]

Since there is a double root at \( x=1 \), we set \( e = \delta^2 \) and formally expand \( x \) in powers of \( \delta = e^{\frac{1}{2}} \):

\[
x[\delta] = x_0 + \text{Sum}[x_n \delta^n, \{n, 3\}] + O[\delta]^4
\]

\[
x_0 + x_1 \delta + x_2 \delta^2 + x_3 \delta^3 + O[\delta]^4
\]

Substituting this formal expansion into the equation (note \( e = \delta^2 \)) gives

\[
(1 - e) x[\delta]^2 + 2 (1 + e) x[\delta] + 1 = 0 / . e \rightarrow \delta^2
\]

\[
(1 - 2 x_0 + x_0^2) + (-2 x_1 + 2 x_0 x_1) \delta + \\
(-2 x_0 - x_0^2 + x_1^2 - 2 x_2 + 2 x_0 x_2) \delta^2 + \\
(-2 x_1 - 2 x_0 x_1 + 2 x_2 x_1 - 2 x_3 + 2 x_0 x_3) \delta^3 + \\
(-x_0^2 - 2 x_2 - 2 x_0 x_2) \delta^4 + O[\delta]^5 = 0
\]

To simplify this expression let us now substitute \( x_0=1 \), which is the (double) root we’re focusing on:

\[
(1 - e) x[\delta]^2 - 2 (1 + e) x[\delta] + 1 = 0 / . e \rightarrow \delta^2 / . x_0 \rightarrow 1
\]

\[
(3 + x_1^2) \delta^2 + (-4 x_1 + 2 x_1 x_2) \delta^3 + (-x_0^2 - 4 x_2) \delta^4 + O[\delta]^5 = 0
\]

Since we only substituted in an expression correct to \( O[\delta]^3 \), we can only trust the terms to \( O[\delta]^3 \) in the above expression. These however will give us the coefficients we’re after. Equating coefficients at \( O[\delta]^2 \) yields

\[
\text{Solve}[(3 + x_1^2) = 0]
\]

\[
\Rightarrow \{x_1 \rightarrow -\sqrt{3}, \ x_1 \rightarrow \sqrt{3}\}
\]
Equating coefficients at $O(\epsilon^3)$ and using the $x_1$ values:

\[
\text{Solve}\left[\left((-4 x_1 + 2 x_2) / \{x_1 \to \pm \sqrt{3}\}\right) = 0\right]
\]

\[
\{(x_2 \to 2)\}
\]

Hence there are two expansion sequences for the two roots, which bifurcate from the single root corresponding to $\epsilon=0$:

\[
x[\epsilon] / \{x_2 \to 2, x_1 \to -\sqrt{3}, x_0 \to 1\}
\]

\[
x[\epsilon] / \{x_2 \to 2, x_1 \to \sqrt{3}, x_0 \to 1\}
\]

\[
1 - \sqrt{3} \epsilon + 2 \epsilon^2 + x_3 \epsilon^3 + O[\epsilon]^4
\]

\[
1 + \sqrt{3} \epsilon + 2 \epsilon^2 + x_3 \epsilon^3 + O[\epsilon]^4
\]

(b) Since $\epsilon$ is multiplying the highest degree term, we have a singular perturbation problem. First we solve the given equation with $\epsilon=0$:

\[
\text{P}_3 [x_, \epsilon_] := \epsilon^2 x^3 + x^2 + 2 x + \epsilon;
\]

\[
\text{Solve}[\text{P}_3 [x, \epsilon] = 0 / \{\epsilon \to 0, x\}]
\]

\[
\{(x \to -2), (x \to 0)\}
\]

We see that we only have two roots (one is lost when we set $\epsilon=0$). For these two roots formally expand in powers of $\epsilon$:

\[
x[\epsilon_] = x_0 + \text{Sum}[x_n \epsilon^n, \{n, 2\}] + O[\epsilon]^3
\]

\[
x_0 + x_1 \epsilon + x_2 \epsilon^2 + O[\epsilon]^3
\]

Substituting this into the equation gives

\[
\epsilon^2 x[\epsilon]^3 + x[\epsilon]^2 + 2 x[\epsilon] + \epsilon = 0
\]

\[
(2 x_0 + x_0^2) + (1 + 2 x_1 + 2 x_0 x_1) \epsilon +
(2 x_0 + x_0^2 + 2 x_2 + 2 x_0 x_2) \epsilon^2 + O[\epsilon]^3 = 0
\]

Equating coefficients of powers of $\epsilon$ yields simultaneous equations relating $x_0, x_1, \ldots$ (the symbol ‘\&’ means logical ‘AND’):

\[
\text{LogicalExpand[]}[
2 x_0 + x_0^2 = 0 \&\& 1 + 2 x_1 + 2 x_0 x_1 = 0 \&\& x_0^3 + x_1^2 + 2 x_2 + 2 x_0 x_2 = 0
\]

And so solving these relations for $x_0, x_1, \ldots$ gives:

\[
\text{Solve}[]
\]
Hence there are two expansion sequences for the two roots close to \(x_0 = -2\) & \(x_0 = 0\):

\[
\begin{align*}
\{x_2 \to -\frac{31}{8}, x_1 \to \frac{1}{2}, x_0 \to -2\}, & \quad \{x_2 \to -\frac{1}{6}, x_1 \to -\frac{1}{2}, x_0 \to 0\}
\end{align*}
\]

Now let us consider the third root. which we lost. We rescale \(x\) so that this root remains \(O[1]\). Rescaling \(x \to y/e^2\) gives

\[
In[77]:= R_3[y, e] := e^2 y^3 + x^3 + 2 x + e, \quad x \to y/e^2;
\]

\[
\text{Simplify}[e^4 R_3[y, e]]
\]

\[y^2 + y^3 + 2 y e^2 + e^5\]

When \(e=0\):

\[
\text{Solve}[\text{Simplify}[e^4 R_3[y, e]] = 0] / . e \to 0, y
\]

\[{\{y \to -1\}, \{y \to 0\}, \{y \to 0\}}\]

Hence there are three roots. The double root \(y=0\) actually corresponds to the two roots already found. Hence we focus on the root \(y=-1\) and proceed as before:

\[
y[e_\_] = y_0 + \text{Sum}[y_n e^n, \{n, 2\}] + O[e]^3
\]

\[
y_0 + y_1 e + y_2 e^2 + O[e]^3
\]

Substituting this into the equation then gives

\[
y[e]^2 + y[e]^3 + 2 y[e] e^2 + e^5 = 0
\]

\[
(y_0^2 + y_1^2) + (2 y_0 y_1 + 3 y_0^2 y_1) e + \\
\left(2 y_0 + y_1^2 + 2 y_0 y_2 + y_0^3 \left(\frac{3 y_1^2}{y_0^2} + \frac{3 y_2}{y_0}\right)\right) e^2 + O[e]^3 = 0
\]

Equating coefficients of powers of \(e\) yields simultaneous equations relating \(y_0, y_1, \ldots\)

\[
\text{LogicalExpand}[\%]
\]
And so solving these relations for \( y_0, y_1, \ldots \) gives:

\[
\text{Solve}[\% \& \ y_0 \to -1] \\
\{(y_2 \to 2, y_1 \to 0)\}
\]

Hence the expansion sequence for the third root, rescaling back to the old variable is:

\[
x[\varepsilon] = y[\varepsilon] / \varepsilon^2 / \{y_2 \to 2, y_1 \to 0, y_0 \to -1\}
\]

\[
-\frac{1}{\varepsilon^2} + 2 + O[\varepsilon]^1
\]

**Question 3**

(a)

\[
f[x] = \sum_{n=0}^{\infty} a_n \varphi_n[x] \text{ as } x \to x_0, \text{ provided } \forall N
\]

(i) \( \frac{\varphi_{n+1}[x]}{\varphi_n[x]} \to 0 \text{ as } x \to x_0; \)

(ii) \( f[x] = \sum_{n=0}^{N} a_n \varphi_n[x] + R_N[x] \) with \( R_N[x] = o(\varphi_N[x]) \) as \( x \to x_0 \).

(b) To obtain an asymptotic series for \( \text{Erf}[x] \) for large \( x \), we first rewrite it in the form

\[
\text{Erf}[x] := 1 - \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} \, dt.
\]

Integrate by parts successively the integral term on the right:

\[
\int_{x}^{\infty} e^{-t^2} \, dt = \int_{x}^{\infty} \frac{1}{2t} \, d(e^{-t^2}) = \frac{e^{-x^2}}{2x} - \int_{x}^{\infty} \frac{e^{-t^2}}{2t} \, dt = \ldots.
\]

One further integration by parts, and then substituting the result back into our modified \( \text{Erf}[x] \) expression reveals the asymptotic series

\[
\text{Series}[\text{Simplify}[\text{Erf}[x], x > 0], \{x, \infty, 3\}]
\]
\[ 1 + e^{-x^2} \left( \frac{1}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} + O\left( \frac{1}{x} \right)^4 \right) \]

(c)

\[ \text{IM}[\_\_] := \int_{-\pi/4}^{\pi/4} e^{x \cos(t)} \cosh(t) \, dt; \]

This is in a form to which we can directly apply Laplace’s method; we notice that the function in the exponent \( g(t) = -C\cos(t) \), has a local maximum, which is also the global maximum within the range of integration, at \( t=0 \). Close to \( t=0 \), the Taylor series approximations for \( g(t) \), up to terms including \( O(t^2) \), and \( \cosh(t) \), up to the zero order, are

\[ g(t) = 1 - \frac{t^2}{2} + O(t^4) \quad \text{and} \quad \cosh(t) = 1. \]

Hence we collapse our integration endpoints to a small neighbourhood around \( t=0 \), thereby only introducing exponentially small errors, and replace \( g(t) \) and \( \cosh(t) \) by these approximations. Then we extend the limits of integration by infinite ones (again introducing exponentially small errors) so that

\[ I[\_] = \int_{-\infty}^{\infty} e^{x \left( \frac{1}{2} - \frac{t^2}{2} \right)} \, dt = e^x \int_{-\infty}^{\infty} e^{-x \frac{t^2}{2}} \, dt \]

Using the substitution \( v^2 = x \frac{t^2}{2} \), and, from the table, that \( \int_{-\infty}^{\infty} e^{-v^2} \, dv = \sqrt{\pi} \), we get

\[ I[\_] = e^x \left( \sqrt{2\pi} \frac{1}{x} + O\left( \frac{1}{x^3/2} \right) \right) \quad \text{as} \quad x \to \infty. \]

(d)

\[ \text{SF}[\_\_] := \int_{0}^{1} t \sin[x(t^3-t)] \, dt = \text{Im}\left[ \int_{0}^{1} t e^{ix(t^2-t)} \, dt \right]. \]

The function in the exponent \( g(t) = t^3 - t \), has stationary points where

\[ \frac{d}{dt} (t^3 - t) = 3t^2 - 1 = 0 \quad \Rightarrow \quad t = \pm \frac{1}{\sqrt{3}}. \]
The positive stationary point is within the range of integration, and

$$\frac{d^2}{dt^2} (t^3 - t) = 6 t = +2 \sqrt{3} \quad \text{at} \quad t = + \frac{1}{\sqrt{3}}.$$

Hence for $t$ close to $t = +\frac{1}{\sqrt{3}}$,

$$g[t] = -\frac{2}{3 \sqrt{3}} + \sqrt{3} \left( t - \frac{1}{\sqrt{3}} \right)^2.$$

Thus collapsing our integration endpoints to a small neighbourhood around $t=1/\sqrt{3}$, substituting the last two approximations mentioned and then extending the limits of integration to infinity (all this only only introduces asymptotically smaller terms), we get

$$\text{SF}[x] = -\frac{1}{\sqrt{3}} \text{Im}[e^{-\frac{2}{\sqrt{3}}} \int_{-\infty}^{\infty} e^{\sqrt{3} i x (t - \frac{1}{\sqrt{3}})^2} \, dt].$$

Using the substitution $v^2 = \sqrt{3} x (t - \frac{1}{\sqrt{3}})^2$ and, from the table, that $\int_{-\infty}^{\infty} e^{i v^2} \, dv = \sqrt{\pi} e^{i n/4}$, we see that

$$\text{SF}[x] = -\frac{\sqrt{\pi}}{3^{3/4} \sqrt{x}} \text{Im}[e^{-\frac{2}{\sqrt{3}}} e^{i n/4}] = \frac{\sqrt{\pi}}{3^{3/4} \sqrt{x}} \sin[\frac{\pi}{4} - \frac{2 x}{3 \sqrt{3}}].$$

**Question 4**

For an arbitrary region $\Omega$ in $\mathbb{R}^3$, the flux of heat leaving the surface per unit area is given by

$$q = -K(T) \nabla T.$$

Hence the total flux of heat leaving the region $\Omega$ is

$$\int \int q \cdot n \, dA$$

where $n$ is the outer unit normal to $\partial \Omega$. The total amount of heat energy in $\Omega$ is

$$\int \int \int c \rho T \, dv.$$
where $C$ is the specific heat capacity (constant) and $\rho$ is the density. Since, rate of decrease of total energy equals the total flux of heat from the surface

$$\Rightarrow -\frac{d}{dt} \iiint C \rho T \, dV = \iint_{\partial V} q \cdot n \, dA .$$

Using the divergence theorem this becomes

$$\Rightarrow -\frac{d}{dt} \iiint C \rho T \, dV = \iiint \nabla \cdot q \, dV .$$

Since $C$ and $\rho$ are constant

$$\Rightarrow \frac{d}{dt} \iiint C \rho T \, dV = C \rho \frac{d}{dt} \iiint T \, dV ,$$

and using that $\Omega$ is arbitrary

$$\Rightarrow C \rho \frac{\partial T}{\partial t} = -\nabla \cdot q .$$

Using our expression for $q$,

$$\Rightarrow \frac{\partial T}{\partial t} = \nabla \cdot \left( \frac{K(T)}{C \rho} \nabla T \right) .$$