Asymptotic Analysis F13YQ1: Final Exam 2003 Model Answers

Question 1

(a)

(i) \( f(t) = O(w(t)) \) as \( t \to t_0 \) \( \Leftrightarrow \) 
\[ \exists A > 0, B > 0 \text{ s.t. } |f(t)| < B |w(t)| \text{ for } |t - t_0| < A. \]

(ii) \( g(t) = o(w(t)) \) as \( t \to t_0 \) \( \Leftrightarrow \) 
\[ \frac{g(t)}{w(t)} \to 0 \text{ as } t \to t_0. \]

(iii) \( g(t) = w(t) \) as \( t \to t_0 \) \( \Leftrightarrow \) 
\[ \frac{g(t)}{w(t)} \to 1 \text{ as } t \to t_0. \]

(b) Since \( x = O[t^2] \) and \( y = o[t^6] \) as \( t \to 0 \), we know that
\[ \exists A > 0, B > 0 \text{ s.t. } |x(t)| < B |t^2| \text{ for } |t| < A, \]
where \( \frac{y(t)}{t^6} \to 0 \text{ as } t \to 0. \)

\[ \Rightarrow \quad \left| \frac{xy^2}{t^5} \right| = \left| \frac{x}{t^2} \frac{y^2}{t^3} \right| < B \left( \left| \frac{y}{t^6} \right| \right)^{\frac{1}{2}} \]
for \( |t| < A \). The RHS \( \to 0 \) as \( t \to 0 \), giving the required result.

(c) Using the Taylor series for \( \sin(x) \):
\[ \sin(x) = x - \frac{x^3}{6} + O[x]^5 \Rightarrow \frac{\sin^2[x]}{x^2} - 1 \text{ as } x \to 0. \]

i.e. \( \exists A > 0 \text{ s.t. } \forall |x| < A, \frac{1}{2} < \left| \frac{\sin^2[x]}{x^2} \right| \leq 2. \)

Consider
\[ \left| \frac{\sin^2[x]}{x^p} \right| = \left| \frac{\sin^2[x]}{x^2} \cdot \frac{1}{x^{p-2}} \right|. \]

\( p < 2 \Rightarrow \left| \frac{\sin^2[x]}{x^p} \right| \leq 2A^{2-p} \)
where the lower bound on the right is unbounded as \( x \to 0 \). Hence the statement is true for \( p < 2 \).

(d)

\[
\begin{align*}
 f(x) & = \sum_{n=0}^{\infty} a_n \varphi_n(x) \quad \text{as} \quad x \to x_0, \quad \text{provided} \ \forall \ N \\
 \frac{\varphi_{n+1}(x)}{\varphi_n(x)} & \to 0 \quad \text{as} \quad x \to x_0; \\
 (i) \quad f(x) & = \sum_{n=0}^{N} a_n \varphi_n(x) + R_N(x), \quad R_N(x) = O[\varphi_N(x)] \quad \text{as} \quad x \to x_0.
\end{align*}
\]

(e) Suppose that in a neighbourhood of \( x_0 \),

\[
S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n < \infty
\]

The remainder after \( N \) terms is

\[
R_N(x) = S(x) - \sum_{n=0}^{N} a_n (x - x_0)^n = \sum_{n=N+1}^{\infty} a_n (x - x_0)^n
\]

The result immediately follows since

\[
\forall N, \quad \frac{R_N(x)}{(x - x_0)^{N+1}} \to a_{N+1} \quad \text{as} \quad x \to x_0
\]

\[
\quad \Rightarrow R_N(x) = O[(x - x_0)^{N+1}] = O[(x - x_0)^N] \quad \text{as} \quad x \to x_0.
\]

**Question 2**

(a) First we solve the given cubic equation with \( \varepsilon = 0 \) (note that it is a singular perturbation problem):

\[
\text{Solve}[\varepsilon x^3 - x^2 + 2x - 1 = 0 \ / \ . \ . \ . \ \varepsilon \to 0, \ x] \quad \Leftrightarrow \quad \{x \to 1\}, \ \{x \to 1\}
\]

Since there is a double root at \( x = 1 \), we set \( \varepsilon = \delta^2 \) and formally expand \( x \) in powers of \( \delta = \varepsilon^{\frac{1}{2}} \):

\[
x[\delta] := 1 + x_1 \delta + x_2 \delta^2 + x_3 \delta^3 + O[\delta]^4
\]

Substituting this formal expansion into the equation gives
\[
\delta^2 x[\delta]^3 - x[\delta]^2 + 2x[\delta] - 1 = 0
\]
\[
\Leftrightarrow (1 - x_1^2) \delta^2 + (3x_1 - 2x_1 x_2) \delta^3 + O[\delta]^4 = 0
\]
Since we only substituted in an expression including terms of \(O[\delta]^3\), we can only trust the terms to \(O[\delta]^3\) in the above expression. These however will give us the coefficients we’re after. Equating coefficients at \(O[\delta]^2\) yields

\[
\text{Solve}\[(1 - x_1^2) = 0\] \Leftrightarrow \{(x_1 \to -1), (x_1 \to 1)\}
\]
Equating coefficients at \(O[\delta]^3\) and using the \(x_1\) values:

\[
\text{Solve}\[((3x_1 - 2x_1 x_2) / \{x_1 \to \pm 1\}) = 0\] \Leftrightarrow \{(x_2 \to \frac{3}{2})\}
\]
Hence there are two expansion sequences for the two roots close to +1:

\[
x[\varepsilon] = 1 - \sqrt{\varepsilon} + \frac{3\sqrt{\varepsilon}}{2} + O[\sqrt{\varepsilon}]^3
\]
\[
x[\varepsilon] = 1 + \sqrt{\varepsilon} + \frac{3\sqrt{\varepsilon}}{2} + O[\sqrt{\varepsilon}]^3
\]
Now let us consider the third singular root. We rescale \(x\) so that this root remains \(O[1]\). Rescaling \(x \to y/\varepsilon\) gives

\[
\text{Simplify}[\varepsilon^2 (\varepsilon x^3 - x^2 + 2x - 1 / \{x \to x/\varepsilon\})] = x^3 - X^2 + 2X\varepsilon - \varepsilon^2
\]
When \(\varepsilon = 0\):

\[
\text{Solve}[\{(X^3 - X^2 + 2X\varepsilon - \varepsilon^2 = 0) / \{\varepsilon \to 0, X\}\] \Leftrightarrow \{(X \to 0), (X \to 0), (X \to 1)\}
\]
Hence there are three roots. The double root \(X=0\) actually corresponds to the two roots already found. Hence we focus on the root \(X=1\) and proceed as before:

\[
X[\varepsilon_] := 1 + x_1 \varepsilon + x_2 \varepsilon^2 + O[\varepsilon]^3
\]
Substituting this into the equation then gives

\[
X[\varepsilon]^3 - X[\varepsilon]^2 + 2X[\varepsilon] \varepsilon - \varepsilon^2 = 0
\]
\[
\Leftrightarrow (2 + x_1) \varepsilon + (-1 + 2x_1 + 2X_1^2 + x_2) \varepsilon^2 + O[\varepsilon]^3 = 0
\]
Equating coefficients of powers of \(\varepsilon \Rightarrow\)
\[ \varepsilon : \quad X_1 = -2 \]

\[ \varepsilon^2 : \quad \text{Solve}\{ (-1 + 2 X_1 + 2 X_1^2 + X_1 = 0) / . \, X_1 \to -2] \Leftrightarrow \{X_2 \to -3\} \]

Hence the expansion sequence for the third root, rescaling back to the old variable is:

\[ x[\varepsilon] = \frac{1}{\varepsilon} - 2 - 3 \varepsilon + O[\varepsilon]^2 \]

(b) Graphing \( \varepsilon e^{y^2} \) and \( y \):

\[
\text{Plot}\left[ \{\varepsilon e^{y^2} / . \{\varepsilon \to 0.2\}, y\}, \{y, 0, 2\} \right]
\]

(i) For the smaller root we solve the transcendental equation with \( \varepsilon = 0 \Rightarrow y = 0 \).

We formally expand \( y(\varepsilon) \) in powers of \( \varepsilon \), and since \( y/e^{y^2} \) is odd, we need only expand in odd powers of \( \varepsilon \):

\[ y[\varepsilon_] := y_1 \, \varepsilon + y_3 \, \varepsilon^3 + y_5 \, \varepsilon^5 + O[\varepsilon]^7 \]

Substituting this formal expansion into the equation gives

\[
\text{Series}\left[ e \, \text{Exp}\{(y[\varepsilon])^2\} - y[\varepsilon], \{\varepsilon, 0, 5\} \right] =
(1 - y_1) \, \varepsilon + (y_3^2 - y_3) \, \varepsilon^3 + \left( \frac{1}{2} \, y_1^4 + 2 \, y_1 \, y_3 - y_3 \right) \, \varepsilon^5 + O[\varepsilon]^6
\]

Equating coefficients at \( O[\varepsilon] \) and \( O[\varepsilon]^3 \) yields

\[ y_1 = 1 \quad \& \quad y_3 = 1. \]
Equating coefficients at $O(\varepsilon)^5 \Rightarrow$

$$\text{Solve} \left[ \left( \frac{1}{2} y_1^4 + 2 y_1 y_3 - y_5 \right) / \{ y_1 \rightarrow 1, \ y_3 \rightarrow 1 \} = 0 \right] \ \Rightarrow$$

$$\{ \{ y_5 \rightarrow \frac{5}{2} \} \}$$

Hence the small solution has the asymptotic approximation

$$y[\varepsilon] = e + e^3 + \frac{5 \varepsilon^5}{2} + O[\varepsilon]^6$$

(ii) For the larger singular and positive root

$$e e^{y^2} = y \ \Leftrightarrow \ y^2 = \ln[1/e] + \ln[y] \ \Leftrightarrow \ y = +\sqrt{\ln[1/e] + \ln[y]}.$$

Since $y \gg \ln[y]$ for $y$ large, a suitable iteration scheme is

$$y_{n+1} = +\sqrt{\ln[1/e] + \ln[y_n]}, \quad y_0 = +\sqrt{\ln[1/e]}.$$

Note that

$$\frac{d}{dt} \left( +\sqrt{\ln[1/e] + \ln[y]} \right) = \frac{1}{2} \frac{1}{y} \frac{1}{+\sqrt{\ln[1/e] + \ln[y]}}$$

$$\Rightarrow \frac{d}{dt} \left( +\sqrt{\ln[1/e] + \ln[y]} \right) = \frac{1}{2 \ln[1/e]} \ll 1,$$

for $y$ close to the large singular root.

Iterating

$$\Rightarrow y_1 = +\sqrt{\ln[1/e] + \frac{1}{2} \ln[\ln[1/e]]}$$

$$\Rightarrow y_1 = +\sqrt{\ln[1/e]} \left( \sqrt{1 + \frac{1}{2} \frac{\ln[\ln[1/e]]}{\ln[1/e]}} \right)$$

$$\Rightarrow y_1 = +\sqrt{\ln[1/e]} \left( 1 + \frac{1}{4} \frac{\ln[\ln[1/e]]}{\ln[1/e]} + \ldots \right)$$

i.e. $y[\varepsilon] = +\sqrt{\ln[1/e]} + \frac{1}{4} \frac{\ln[\ln[1/e]]}{\sqrt{\ln[1/e]}}.$

**Question 3**

(a) Integrating by parts successively
The expansion sequence is clearly an asymptotic sequence.

Now consider

$$
\forall N : \quad |R_{tN}[x]| = \left| N! \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt \right| \leq N! \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt
$$

$$
\leq \frac{N!}{x^{N+1}} \int_x^\infty e^{-t} dt = e^{-x} \frac{N!}{x^{N+1}}.
$$

$$
\Rightarrow \quad \left| R_{tN}[x] \right| \leq e^{-x} \frac{N!}{x^{N+1}} \leq e^{-x} (N-1)!/x^N \xrightarrow{N \to 0} 0 \text{ as } x \to \infty.
$$

(b)

$$
LM[x] := \int_0^\infty e^{-x \cosh t} \cosh[\nu t] dt;
$$

Directly applying Laplace’s method; we notice that \( \varphi[t] = -\cosh[t] \) has a local and global maximum within the range of integration at \( t=0 \). Close to \( t=0 \), Taylor series approximations for \( \varphi[t] \), up to terms including \( O[t^2] \), and \( \cosh[\nu t] \) to zero order, are

$$
\varphi[t] = -1 - \frac{t^2}{2} + O[t^4] \quad \text{and} \quad \cosh[\nu t] = 1.
$$

Hence we collapse our range of integration to the small neighbourhood including \( t=0 \) and just to the right of \( t=0 \), thereby only introducing exponentially small errors, and replace \( \varphi[t] \) and \( \cosh[\nu t] \) by these approximations. Then we extend the limits of integration to the semi-infinite range \( [0, \infty) \) (again introducing exponentially small errors) so that

$$
I[x] = \int_0^\infty e^{x (-1 - \frac{t^2}{2})} dt = e^{-x} \int_0^\infty e^{-x \frac{t^2}{2}} dt
$$

Using the substitution \( \nu^2 = x \frac{t^2}{2} \) and that \( \int_0^\infty e^{-\nu^2} d\nu = \sqrt{\pi}/2 \)
\[
\Rightarrow I[x] = e^{-x} \left( \sqrt{\frac{\pi}{2x}} + O\left[\frac{1}{x^{3/2}}\right]\right) \text{ as } x \to \infty.
\]

(c) 

\[
SF[x_] := \int_{-\pi/2}^{\pi/2} \sin[x \cos[t]] \, dt = \text{Im} \left[ \int_{-\pi/2}^{\pi/2} e^{i x \cos[t]} \, dt \right].
\]

The function in the exponent \( \psi[t] = \cos[t] \) has stationary points where

\[
\frac{d}{dt} (\cos[t]) = -\sin[t] = 0 \iff t = 0, \pm \pi, \pm 2\pi, ...
\]

The stationary point within the range of integration is at \( t=0 \), and close to \( t=0 \):

\[
\cos[t] = 1 - \frac{t^2}{2}
\]

Thus collapsing our integration endpoints to a small neighbourhood around \( t=0 \), substituting the last approximation mentioned and then extending the limits of integration to infinity (all this only introduces asymptotically smaller terms), we get

\[
SF[x] = \text{Im} \left[ e^{ix} \int_{-\infty}^{\infty} e^{-ix \frac{v^2}{2}} \, dv \right]
\]

The substitution \( v^2 = x \frac{v^2}{2} \) and that \( \int_{-\infty}^{\infty} e^{-i v^2} \, dv = \sqrt{\pi} \, e^{-i \pi/4} \)

\[
\Rightarrow SF[x] = \sqrt{\frac{2\pi}{x}} \, \text{Im} \left[ e^{ix - \frac{\pi}{4}} \right] = \sqrt{\frac{2\pi}{x}} \, \sin \left[ x - \frac{\pi}{4} \right]
\]

**Question 4**

(a) For an arbitrary region \( \Omega \) in \( \mathbb{R}^3 \), the flux of material leaving the surface per unit area is

\[
q = \rho \, v = -\rho \, k \, v \nabla p
\]

Hence the total flux of material leaving the region \( \Omega \) is

\[
\int_{\partial \Omega} q \cdot n \, dA = \int_{\partial \Omega} q \cdot n \, dA
\]
where $n$ is the outer unit normal to $\partial \Omega$. The total amount of material in $\Omega$ is

$$\iiint_\Omega \rho \, dV.$$ 

Since, rate of decrease of total material equals the total flux of material through the surface

$$\Rightarrow -\frac{d}{dt} \iiint_\Omega \rho \, dV = \iint_{\partial \Omega} \mathbf{q} \cdot \mathbf{n} \, dA.$$

Using the divergence theorem this becomes

$$\Rightarrow -\frac{d}{dt} \iiint_\Omega \rho \, dV = \iiint_{\Omega} \nabla \cdot \mathbf{q} \, dV.$$

Using that $\Omega$ is arbitrary

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{q}.$$ 

Using our expression for $\mathbf{q}$,

$$\Rightarrow \frac{\partial \rho}{\partial t} = k \nabla \cdot (\rho \nabla p).$$

(b)

(i) Pressure proportional to a power, $n$, of the density

$$\Rightarrow p = c \rho^n$$ 

for some constant $c$ and substituting this into the PDE above

$$\Rightarrow \frac{\partial \rho}{\partial t} = k \nabla \cdot (\rho \nabla (c \rho^n)) = kc \nabla \cdot (\rho \nabla (\rho^n))$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = kc \nabla \cdot (\rho n \rho^{n-1} \nabla \rho) = K \nabla \cdot (\nabla (\rho^n \nabla \rho))$$

where $K=kcn$.

(ii) Set

$$u = K^{1/n} \rho \quad \Rightarrow \quad u^n = K \rho^n$$

$$\Rightarrow \quad K^{-1/n} \frac{\partial u}{\partial t} = K \nabla \cdot (K^{-1/n} u^n \nabla (K^{-1/n} u)) \quad \Rightarrow \quad \frac{\partial u}{\partial t} = \nabla \cdot (u^n \nabla (u))$$