Suppose we are asked to find approximate power series solutions to the algebraic equation:

\[ F(u(\epsilon); \epsilon) = 0. \]  

(1)

We suppose \( \epsilon \) to be a small parameter. If the problem involves a large parameter \( L \gg 1 \) then set \( L = 1/\epsilon \); some rescaling of the problem may be necessary. Also assume that there exists a finite solution \( u(\epsilon) \) to (1) when \( \epsilon \) is small, that is ‘close’ to the solution of interest \( u_0 \) which solves

\[ F(u_0; 0) = 0. \]  

(2)

Lastly, we shall also suppose that \( F \) has a Taylor series expansion about \( u_0 \).

**Expansion method.** For small \( \epsilon \), to find an approximate solution of (1) that is close to \( u_0 \), expand \( u(\epsilon) \) as a formal power series in \( \epsilon \):

\[
 u(\epsilon) = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots.
\]  

(3)

Here \( u_0 \) is the solution to (2). As we have seen, it makes sense to determine the order of the root \( u_0 \) to (2), as this will determine the trial expansion we should use—for example if \( u_0 \) is a double root of (2) then we should expand in powers of \( \epsilon^{1/2} \). For the moment, suppose \( u_0 \) is a simple root.

The idea is to first expand \( F \) itself as a Taylor series (if needed):

\[
 F(u(\epsilon); \epsilon) = F(u_0; 0) + \epsilon \left( F_u(u_0; 0)u'(0) + F_\epsilon(u_0; 0) \right) \\
 + \frac{\epsilon^2}{2!} \left( F_{uu}(u_0; 0)(u'(0))^2 + 2F_{ue}(u_0; 0)u'(0) \\
 + F_{ee}(u_0; 0) + F_u(u_0; 0)u''(0) \right) + \cdots.
\]

Now using that for the power series expansion (3) we have that \( u_1 = u'(0) \), \( u_2 = u''(0)/2! \), \ldots etc., substituting these identities into (1) we get

\[
 F(u_0; 0) + \epsilon \left( F_u(u_0; 0)u_1 + F_\epsilon(u_0; 0) \right) \\
 + \frac{\epsilon^2}{2!} \left( F_{uu}(u_0; 0)u_1^2 + 2F_{ue}(u_0; 0)u_1 + F_{ee}(u_0; 0) + 2F_u(u_0; 0)u_2 \right) + \cdots = 0.
\]  

(4)

Now we equate the coefficients of the powers of \( \epsilon \) in (4) to obtain \( u_1, u_2, \ldots \) etc., and thus the various orders of approximations (3) to \( u(\epsilon) \).

**Warning:** Tread carefully with this procedure!

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Formally, we get:

\[ \epsilon^0 : F(u_0; 0) = 0 \quad \text{(automatically satisfied by choice of } u_0) ; \]

\[ \epsilon^1 : F_u(u_0; 0)u_1 + F_\epsilon(u_0; 0) = 0 \quad \Rightarrow \quad u_1 = -\frac{F_\epsilon(u_0; 0)}{F_u(u_0; 0)} ; \]

\[ \epsilon^2 : F_{uu}(u_0; 0)u_1^2 + 2F_{u\epsilon}(u_0; 0)u_1 + F_{\epsilon\epsilon}(u_0; 0) + 2F_u(u_0; 0)u_2 = 0 ; \]

and so forth. The equation at order \( \epsilon^2 \) gives us that

\[ u_2 = -\frac{F_{uu}(u_0; 0)u_1^2 + 2F_{u\epsilon}(u_0; 0)u_1 + F_{\epsilon\epsilon}(u_0; 0)}{2F_u(u_0; 0)} . \]

Hence formally we can determine \( u_1, u_2, u_3, \ldots \) etc. However we see that we will have problems if \( F_u(u_0; 0) = 0 \), which would occur if \( u_0 \) was a root of \( (2) \) of order two or more. If \( F_u(u_0; 0) = 0 \) and \( F_{uu}(u_0; 0) \neq 0 \) then \( u_0 \) is a double root of \( (2) \) and instead of \( (3) \) we should have instead posed a power series expansion for \( u(\epsilon) \) in powers of \( \epsilon^\frac{1}{2} \). More generally, suppose \( \partial_u F(u_0; 0) = \cdots = \partial_u^{n-1} F(u_0; 0) = 0 \) and \( \partial_u^n F(u_0; 0) \neq 0 \), then we should formally pose a power series expansion for \( u(\epsilon) \) in powers of \( \epsilon^\frac{1}{n} \).

Note that we can generalize some of these ideas to:

- Systems of algebraic equations: \( F(u; \epsilon) = 0 \);
- Ordinary differential equations: \( F(x, u(x), u'(x), \ldots; \epsilon) = 0 \);
- Partial differential equations: \( F(x, u(x), \nabla u(x), \ldots; \epsilon) = 0 \).

**Iteration method.** The iteration method first involves rearranging the algebraic equation (1) into the form

\[ u(\epsilon) = G(u(\epsilon); \epsilon) . \]

(5)

If the solution \( u_0 \) of (2) is a root of order two or more then we should construct the rearrangement (3) appropriately to naturally reflect how \( u(\epsilon) \) locally scales with \( \epsilon \). It is also essential to ensure that

\[ |G_u(u; \epsilon)| < 1 , \]

for \( u \) close to the true solution \( u(\epsilon) \), for \( \epsilon \) small. We can then adopt the iterative process

\[ u_{n+1}(\epsilon) = G(u_n(\epsilon); \epsilon) , \quad \text{for} \; n \geq 1 , \]

(6)

starting with \( u_0 \).