CONSERVATION LAWS

1. Multivariable calculus

1.1. Divergence theorem (of Gauss). This states that the volume integral in $\Omega$ of the divergence of the vector-valued function $F$ is equal to the total flux of $F$ through the surface $\partial \Omega$ ($\hat{n}$ is the unit outer normal to the surface):

$$\int_{\Omega} \nabla \cdot F \, dV = \int_{\partial \Omega} F \cdot \hat{n} \, dS.$$ 

See Figures 2 and 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Divergence theorem: the total flux of $F$ through the surface $\partial \Omega$ is equal to the volume integral in $\Omega$ of the divergence of $F$. Here $\hat{n}$ is the unit outer normal to the surface.}
\end{figure}

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Figure 2. Flux: for example, for a fluid the amount of material passing through/across the small area patch $dS$ in the surface per unit time, is roughly the density of the material times the volume of the cylinder shown $\approx (v \cdot \hat{n}) dS$, where $v$ is the local velocity of the fluid.
1.2. **Stokes’ theorem.** This states that the integral of the flux of the curl of $F$ through the surface $S$ shown in Figure 3 is equal to the path integral of $F$ round the bounding contour $C$:

$$\int_S (\nabla \times F) \cdot \hat{n} \, dS = \oint_C F \cdot dr.$$ 

![Figure 3](image.png)

**Figure 3.** Stokes’ theorem: the integral of the flux of $\nabla \times F$ through the surface $S$ shown is equal to the path integral of $F$ round the bounding contour $C$ shown.
1.3. **Stokes’ theorem in the plane.** If the vector-valued function $F$ is *conservative* in some simply connected domain $D$, i.e. in $D$ there exists a scalar potential function $\Phi$ such that

$$F = \nabla \Phi,$$

then (see Figure 4)

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr = \Phi(B) - \Phi(A),$$

for any two contours $C_1$ and $C_2$ lying in $D$, with starting point $A$ and end point $B$. In other words, the contour integral of $F$ in such a case is independent of the path chosen and depends only on the endpoints $A$ and $B$.

In particular for a closed contour $C$ lying in $D$, Stokes’ theorem in the plane says:

$$\oint_C F \cdot dr = 0.$$

Recall Cauchy’s theorem from complex analysis: if $f(z)$ is analytic in a closed simply connected domain $D$, then for every simple closed path $C$ in $D$,

$$\oint_C f(z) dz = 0.$$

**Figure 4.** Stokes’ theorem in the plane: for any conservative function $F(r)$ in the simply connected domain $D$, the path integral between any two points $A, B \in D$ is independent of the path chosen. This is also known as the *path independence property* for conservative functions.
2. Heat equation

2.1. Physics. Suppose we have a large slab of material that has been heated. We would like to model how the temperature profile within the material will evolve in time. Let \( \Omega \) be an arbitrary closed subregion of the material slab then we have the following physical definitions:

- If \( \rho \) is the density (mass per unit volume) and \( H \) is the heat content per unit mass, then the total heat content in the region \( \Omega \) is
  \[
  \int_\Omega \rho H \, dV.
  \]
  The heat content per unit mass \( H \) can be expressed in the form
  \[
  H = \int_{T_0}^{T} C(\tau) \, d\tau, \tag{1}
  \]
  where \( C(T) \) is the specific heat capacity at temperature \( T \) and \( T_0 \) is a reference temperature.

- The flow of heat energy out of the region \( \Omega \) is
  \[
  \int_{\partial \Omega} q \cdot n \, dS,
  \]
  where \( q \) is the flux of heat energy (measures the amount and direction of energy transport per unit time) and \( n \) is the unit outer normal to the bounding surface of \( \Omega \), denoted by \( \partial \Omega \).

- Also suppose that heat is being generated internally (from some source) within the material. This could be due to a chemical reaction taking place internally or due to an external electric–magnetic field acting on the slab that is heating it up internally due to hysteresis effects. If heat is being generated at the rate \( Q \) per unit volume (assume we know what \( Q \) is) then the total heat generated within \( \Omega \) is
  \[
  \int_\Omega Q \, dV.
  \]

2.2. Conservation of energy. Conservation of heat energy implies that

\[
\frac{d}{dt} \int_\Omega \rho H \, dV = - \int_{\partial \Omega} q \cdot n \, dS + \int_\Omega Q \, dV. \tag{2}
\]
Provided $\rho$ and $H$ are nice smooth functions then for the left–hand side we can write
\[
\frac{d}{dt} \int_{\Omega} \rho H \, dV = \int_{\Omega} \frac{\partial}{\partial t} (\rho H) \, dV,
\]
while for the surface integral on the right–hand side we can use the divergence theorem giving
\[
\int_{\partial \Omega} q \cdot n \, dS = \int_{\Omega} \nabla \cdot q \, dV.
\]
Substituting these last two expressions back into the conservation relation (2) we get
\[
\int_{\Omega} \left( \frac{\partial (\rho H)}{\partial t} + \nabla \cdot q - Q \right) \, dV = 0.
\]
Now recall that our subregion $\Omega$ was chosen completely arbitrarily and so could have been any closed subregion of the material of any size and so we must have that
\[
\frac{\partial (\rho H)}{\partial t} + \nabla \cdot q - Q = 0.
\]

2.3. Simplifying assumptions.

- If we assume that the density $\rho$ is time independent and also that specific heat capacity $C$ in (1) is constant, then this last equation reduces to
\[
\rho C \frac{\partial T}{\partial t} + \nabla \cdot q - Q = 0. \tag{3}
\]
Recall that we assume we know what $Q$ is, but we do not as yet have an appropriate expression for the heat flux $q$. However Fourier’s Law for heat conduction (in a stationary material) says that
\[
q = -\kappa \nabla T,
\]
where $\kappa = \kappa(T)$ is the thermal conductivity which often depends on the temperature $T$. Basically Fourier’s Law says that heat will flow in the direction of the largest negative temperature gradient (i.e. from hot to cold down the steepest temperature gradient). Substituting this expression for $q$ into (3) we get the general nonlinear heat equation
\[
\rho C \frac{\partial T}{\partial t} = \nabla \cdot (\kappa(T) \nabla T) + Q. \tag{4}
\]
- If $\kappa$ is constant and $Q \equiv 0$ this reduces to the famous linear heat equation
\[
\frac{\partial T}{\partial t} = \mathcal{D} \nabla^2 T, \tag{5}
\]
where the constant $\mathcal{D} = \frac{\kappa}{\rho C}$ is known as the thermal diffusivity.
2.4. **Nonlinear heat equation and rescaling.** Depending on the material, the thermal conductivity $\kappa(T)$ is also often proportional to a power of the temperature $T$, i.e.

$$\kappa(T) = k T^n,$$

where $k$ is a constant of proportionality and $n$ the power growth. In such cases the nonlinear heat equation becomes (with $Q \equiv 0$)

$$\frac{\partial T}{\partial t} = K \nabla \cdot (T^n \nabla T). \tag{6}$$

where $K = \frac{k}{\rho C}$. If we rescale $T$ by setting

$$u = K^{1/n} T,$$

and substitute this into (6) we get

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^n \nabla u), \tag{7}$$

i.e. we have non-dimensionalized the nonlinear heat equation.