(a) From the calculus of variations, we consider perturbations of the minimizing path $q = q(t)$ of the form $q + \epsilon \delta q$ where $\epsilon$ is a real parameter and $\delta q = \delta q(t)$ is any perturbation shape zero at the endpoints $t_1$ and $t_2$. Differentiating $E(q + \epsilon \delta q)$ with respect to $\epsilon$ and requiring the derivative to be zero at $\epsilon = 0$ for all $\delta q$ implies the path $q = q(t)$ that minimizes the total energy $E = E(q)$ necessarily satisfies the Euler Lagrange equations. Here these take the form of Lagrange's equations of motion which are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

for each $j = 1, \ldots, n$.

(b) If we take the Lagrangian to be the kinetic energy

$$L(q, \dot{q}) = \frac{1}{2} \sum_{i,k=1}^{n} g_{ik}(q) \dot{q}_i \dot{q}_k,$$

and substitute this into Lagrange's equations of motion in part (a) we find using the product and chain rules:

$$\frac{\partial L}{\partial q_j} = \frac{1}{2} \sum_{i,k=1}^{n} \frac{\partial g_{ik}}{\partial q_j} \dot{q}_i \dot{q}_k,$$

and

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{1}{2} \sum_{k=1}^{n} g_{jk} \dot{q}_k + \frac{1}{2} \sum_{i=1}^{n} g_{ij} \dot{q}_i$$

$$= \frac{1}{2} \sum_{k=1}^{n} g_{jk} \dot{q}_k + \frac{1}{2} \sum_{k=1}^{n} g_{kj} \dot{q}_k$$

$$= \sum_{k=1}^{n} g_{jk} \dot{q}_k,$$

where in the last step we utilized the symmetry of $g$. Then we see that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = \sum_{k=1}^{n} \frac{d}{dt} \left( g_{jk} \dot{q}_k \right)$$

$$= \sum_{k=1}^{n} \frac{\partial g_{jk}}{\partial t} \dot{q}_k + \sum_{k=1}^{n} \left( g_{jk} \ddot{q}_k \right)$$

$$= \sum_{k, t=1}^{n} \frac{\partial g_{jk}}{\partial q_t} \dot{q}_k \ddot{q}_k + \sum_{k=1}^{n} \left( g_{jk} \ddot{q}_k \right).$$

Substituting the expressions above into Lagrange's equations of motion, using that the summation indices $i$ and $k$ can be relabelled and that $g$ is symmetric, we find

$$\sum_{k=1}^{n} g_{jk} \ddot{q}_k + \sum_{k, t=1}^{n} \left( \frac{\partial g_{jk}}{\partial q_t} - \frac{1}{2} \frac{\partial g_{kt}}{\partial q_j} \right) \dot{q}_k \ddot{q}_t = 0,$$

for each $j = 1, \ldots, n$.  

(c) If \( g^{ij} \) denotes the inverse matrix of \( g_{ij} \) (where \( i, j = 1, \ldots, n \)) so that

\[
\sum_{k=1}^{n} g^{ik} g_{kj} = \delta_{ij},
\]

then we can multiply the geodesic equations from part (b) by \( g^{ij} \) and sum over the index \( j \). This gives

\[
\sum_{j,k=1}^{n} g^{ij} g_{jk} \dot{q}_k + \sum_{j,k,t=1}^{n} g^{ij} \left( \frac{\partial g_{jk}}{\partial q_t} - \frac{1}{2} \frac{\partial g_{jt}}{\partial q_{kj}} \right) \dot{q}_k \dot{q}_t = 0
\]

\[
\Leftrightarrow \quad \ddot{q}_i + \sum_{j,k=1}^{n} g^{ij} \left( \frac{\partial g_{jk}}{\partial q_t} - \frac{1}{2} \frac{\partial g_{jt}}{\partial q_{kj}} \right) \dot{q}_k \dot{q}_t = 0
\]

\[
\Leftrightarrow \quad \ddot{q}_i + \sum_{j,k=1}^{n} \Gamma^i_{jk} \dot{q}_j \dot{q}_k = 0
\]

for each \( i = 1, \ldots, n \), where

\[
\Gamma^i_{jk} := \sum_{t=1}^{n} g^{it} \left( \frac{\partial g_{tj}}{\partial q_k} - \frac{1}{2} \frac{\partial g_{tk}}{\partial q_{kj}} \right).
\]

(d) We can also re-express \( \Gamma^i_{jk} \) in the form shown, as follows

\[
\sum_{j,k=1}^{n} \Gamma^i_{jk} \dot{q}_j \dot{q}_k = \sum_{j,k,t=1}^{n} \left( \frac{\partial g_{tj}}{\partial q_k} - \frac{1}{2} \frac{\partial g_{tk}}{\partial q_{kj}} \right) \dot{q}_j \dot{q}_k
\]

\[
= \sum_{j,k,t=1}^{n} \left( \frac{\partial g_{tj}}{\partial q_k} + \frac{\partial g_{tj}}{\partial q_k} - \frac{\partial g_{tk}}{\partial q_{kj}} \right) \dot{q}_j \dot{q}_k
\]

\[
= \sum_{j,k,t=1}^{n} \frac{1}{2} g^{it} \left( \frac{\partial g_{tj}}{\partial q_k} + \frac{\partial g_{tk}}{\partial q_{kj}} \right) \dot{q}_j \dot{q}_k,
\]

where we have utilised that the summation indices can all be relabelled.
Solution (Central force problem)

\[ L = \frac{1}{2}m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - V(r) \]

\[ \text{K.E.} \]

(a) \[ p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \]

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} \]

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi} \]

No explicit \( \phi \) in \( L \) \( \Rightarrow \) \( p_\phi \) is a constant of motion.

No explicit \( \theta \) in \( L \) \( \Rightarrow \) \( \frac{dH}{dt} = \frac{\partial H}{\partial \dot{r}} \frac{\partial \dot{r}}{\partial t} + \frac{\partial H}{\partial \dot{\theta}} \frac{\partial \dot{\theta}}{\partial t} + \frac{\partial H}{\partial \dot{\phi}} \frac{\partial \dot{\phi}}{\partial t} = 0 \)

so Hamiltonian also a constant of motion.

K.E. is a homogeneous quadratic function of generalized velocities \( \Rightarrow \) Hamiltonian will be total energy.
Solution (central force problem) (cont'd)

(b)  
\[ H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L \]
\[ = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2 \sin^2 \theta} \]
\[ -\frac{1}{2}m \left( \frac{(\dot{r} p_r)^2}{m^2} + r^2 \left( \frac{\dot{\theta} p_\theta}{mr^2} \right)^2 + r^2 \sin^2 \theta \left( \frac{\dot{\phi} p_\phi}{mr^2 \sin \theta} \right)^2 \right) \]
\[ + V(r) \]
\[ = \frac{1}{2} \frac{p_r^2}{m} + \frac{1}{2} \frac{p_\theta^2}{mr^2} + \frac{1}{2} \frac{p_\phi^2}{mr^2 \sin^2 \theta} + V(r) \]

(c) Use Lagrange's equations of motion noting that \( p_\phi = mr^2 \sin^2 \theta \cdot \dot{\phi} \) is constant:
\[ \frac{d}{dt} \left( \frac{2L}{d\dot{\phi}} \right) - \frac{2L}{d\phi} = 0 \]
\[ \frac{d}{dt} \left( \frac{2L}{d\dot{r}} \right) - \frac{2L}{d\dot{r}} = 0 \]
Solution  (Central force problem)  (cont'd)

\[
\frac{d}{dt} (mr^2 \dot{\theta}) = mr^2 \sin \theta \cos \theta \cdot \dot{\theta}^2
\]

\[
\frac{d}{dt} (m \dot{r}) = m \dot{r}^2 + mr \sin^2 \theta \cdot \ddot{\theta} - V(r)
\]

\[
\frac{d}{dt} (r^2 \dot{\theta}) = r^2 \sin \theta \cos \theta \cdot \left(\frac{P_0}{m r^2 \sin^2 \theta}\right)^2
\]

\[
\frac{d}{dt} (r \dot{r}) = r \dot{r}^2 + r \sin^2 \theta \left(\frac{P_0}{m r^2 \sin^2 \theta}\right)^2 \ddot{r} - \frac{1}{m} V(r)
\]

\[
\frac{d}{dt} (r^2 \dot{\theta}) = \frac{P_0^2}{m^2 r^2} \cdot \cot \theta \cdot \csc^3 \theta
\]

\[
\ddot{r} = r \dot{r}^2 + \frac{P_0^2}{m^2 r^3} \csc^2 \theta - \frac{1}{m} V(r).
\]
Solution (Bernoulli’s Theorem: Equ. 3.4.5.11, Note)

(a) Since the flow is stationary and $\rho$ constant and uniform, Euler’s equations

$\mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \left( \frac{P}{\rho} \right) - \nabla \Phi.$

Using the given identity we get

$\frac{1}{2} \nabla (|\mathbf{u}|^2) - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla \left( \frac{P}{\rho} \right) - \nabla \Phi$

$\Rightarrow \nabla \left( \frac{1}{2} |\mathbf{u}|^2 + \frac{P}{\rho} + \Phi \right) = \mathbf{u} \times (\nabla \times \mathbf{u})$

$\Rightarrow \nabla H = \mathbf{u} \times (\nabla \times \mathbf{u}),$

for the quantity $H$ given in the question.

If $\mathbf{x} = \mathbf{x}(s)$ denotes a streamline satisfying $\mathbf{x}'(s) = \mathbf{u}(\mathbf{x}(s))$, then the fundamental theorem of calculus for any $s_1$ and $s_2$ implies

$H(\mathbf{x}(s_2)) - H(\mathbf{x}(s_1)) = \int_{s_1}^{s_2} dH(\mathbf{x}(s))$

$= \int_{s_1}^{s_2} \nabla H \cdot \mathbf{x}'(s) \, ds$

$= \int_{s_1}^{s_2} (\mathbf{u} \times (\nabla \times \mathbf{u})) \cdot \mathbf{u}(\mathbf{x}(s)) \, ds$

$= 0.$

Have used that $(\mathbf{u} \times \mathbf{a}) \cdot \mathbf{u} = 0$ for any $\mathbf{a}$. Since $s_1$ and $s_2$ arbitrary implies $H$ constant on streamlines.
Solution (Torricelli problem) (cont'd)

(b) Fluid is incompressible, hence rate at which fluid leaves punctured hole equals volume rate at which water drops at free surface, i.e.

\[
\left( -\frac{dh}{dt} \right) \cdot A = \frac{dV}{dt} \cdot \frac{a}{A}
\]

rate at which free surface height drops \hspace{2cm} \text{cross-sectional area of drum}

rate at which fluid leaves \hspace{2cm} \text{cross-sectional area of punctured hole}

(ii) Apply Bernoulli's result for a typical streamline as shown in the figure

\[
\frac{1}{2} \left( -\frac{dh}{dt} \right)^2 + \frac{p_0}{\rho} = \frac{1}{2} U^2 + \frac{p_0}{\rho} - gh
\]

Bernoulli quantity at free surface \hspace{2cm} \text{Bernoulli quantity as fluid exits punctured hole}

\[
\Rightarrow \frac{1}{2} \left( \frac{dh}{dt} \right)^2 = \frac{1}{2} U^2 - gh
\]
Solution (Torricelli problem) (cont'd)

(iii) From (i) we have

\[ \left| \frac{dh}{dt} \right| = \frac{\alpha}{A} \cdot U. \]

Assuming that \( \frac{\alpha}{A} \ll 1 \) we see that

\[ g \cdot h = \frac{1}{2} U^2 - \frac{1}{2} \left( \frac{dh}{dt} \right)^2 \]

\[ = \frac{1}{2} U^2 \left( 1 - \left( \frac{\alpha}{A} \right)^2 \right) \]

\[ = \frac{1}{2} U^2 \left( 1 - \left( \frac{\alpha}{A} \right)^2 \right) \]

\[ \sim \frac{1}{2} U^2 \quad \text{for} \quad \frac{\alpha}{A} \ll 1. \]

Hence in the asymptotic limit

\[ gh = \frac{1}{2} U^2 \]

\[ \Rightarrow \quad U = \sqrt{2gh} \]
Solution (Coffee mug)

(a) Euler equations are
\[ \frac{\partial u_r}{\partial t} + (u \cdot \nabla) u_r - \frac{u^2_r}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + f_r, \]
\[ \frac{\partial u_\theta}{\partial t} + (u \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + f_\theta, \]
\[ \frac{\partial u_z}{\partial t} + (u \cdot \nabla) u_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + f_z, \]
with
\[ u \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}. \]
and
\[ \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \]

Given that \( u_r = u_z = 0, \partial / \partial t \equiv 0, \partial / \partial \theta \equiv 0 \) and \( f_r = f_\theta = 0 \) with \( f_z = -g \). Substituting all these into Euler’s equations \( \Rightarrow \)
\[ -\frac{u^2_r}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \]
\[ 0 = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta}, \]
\[ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g. \]

Note \( \nabla \cdot u = 0 \) trivially. Second equation above \( \leftrightarrow p = p(r, z) \).

(b) Using that \( u_\theta = \Omega r \) \( \Rightarrow \)
\[ \frac{\partial p}{\partial r} = \rho \Omega^2 r \quad \leftrightarrow \quad p(r, z) = \frac{1}{2} \rho \Omega^2 r^2 + G(z), \]
where \( G \) an arbitrary function. Substitute this into other equation
\[ \frac{1}{\rho} \frac{\partial p}{\partial z} = -g \quad \leftrightarrow \quad G'(z) = -pg. \]

Hence \( G(z) = -pgz + C \) where \( C \) is an arbitrary constant. \( \Rightarrow \)
\[ p(r, z) = \frac{1}{2} \rho \Omega^2 r^2 - pgz + C. \]

At free surface \( p = P_0 \):
\[ P_0 = \frac{1}{2} \rho \Omega^2 r^2 - pgz + C \]
\( \leftrightarrow \)
\[ z = (\Omega^2 / 2g) r^2 - (C - P_0) / pg. \]

(c) Choose \( C = P_0 \) \( \leftrightarrow \) \( z = (\Omega^2 / 2g) r^2 \). Relative to chosen origin \( O \), bottom of mug at \( z = -z_0 \).
(i) Initially coffee depth $d \implies$ total volume is $\pi a^2 d$.

Incompressibility volume constraint $\implies$

$$\frac{\pi a^2 z_0}{\text{vol under } O} + \int_0^a \frac{\Omega^2 r^2}{2g} \cdot 2\pi r \, dr = \pi a^2 d.$$ 

(ii) Constraint in part (i) $\iff$

$$z_0 + \frac{\Omega^2 a^2}{4g} = d.$$ 

Coffee at edge of mug is at height

$$z_0 + \frac{\Omega^2 a^2}{2g} = d - \frac{\Omega^2 a^2}{4g} + \frac{\Omega^2 a^2}{2g} = d + \frac{\Omega^2 a^2}{4g}.$$ 

Spillage when this is $> h$, i.e. when $\Omega^2 > 4g(h - d)/a^2$. 
Solution (Separation of variables)

(a) Seek solutions of form \( u(x,t) = X(x) \cdot T(t) \). 
Substitute into PDE \( \Rightarrow \)

\[
X(x) \cdot T'(t) = k X''(x) \cdot T(t)
\]

Hence

\[
\frac{X''(x)}{X(x)} = \frac{T'(t)}{k T(t)} = -\lambda
\]

for some constant \( \lambda \). BCS \( \Rightarrow \) \( X(0) = X(L) = 0 \).

General solution to ODE \( X'' = -\lambda X \) is

\[
X(x) = A \sin(\sqrt{\lambda} x) + B \cos(\sqrt{\lambda} x)
\]

BCs \( \Rightarrow B = 0 \) and either \( A = 0 \) or \( \lambda = \frac{n^2 \pi^2}{L^2} \) for \( n \in \mathbb{Z} \).

\( \Rightarrow \) Solution to ODE \( T' = -\lambda k T \) is

\[
T(t) = A_n \exp\left(-\frac{k n^2 \pi^2}{L^2} \cdot t\right)
\]

Hence any function of form

\[
u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n \pi x}{L}\right) e^{-\frac{k n^2 \pi^2}{L^2} t}
\]

satisfies the PDE and BCS.
Solution (Separation of Variables) (cont'd)

Now choose $A_0, A_1, A_2, \ldots$ etc. so that

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right) = 2 \sin \left( \frac{5\pi x}{L} \right)$$

$\forall \ 0 \leq x \leq L$. Hence $A_0 = \ldots = A_4 = A_6 = \ldots = 0$ and $A_5 = 2$.

Hence solution is $u(x,t) = 2 \sin \left( \frac{5\pi x}{L} \right) e^{-\frac{25\pi^2}{L^2}t}$.

(b) Seek solutions of form $u(x,t) = X(x) \cdot T(t)$. 
Substitute into PDE and separate variables $\Rightarrow$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T''(t)}{T(t)} = -\lambda$$

for some constant $\lambda$. B.C. $\Rightarrow X(0) = X(L) = 0$.
Hence as in part (a), $X(x) = A \sin \left( \frac{n\pi x}{L} \right)$
with $\lambda = \frac{n^2\pi^2}{L^2}$, $n \in \mathbb{Z}$. 
Solution (Separation of Variables) (cont'd)

Now consider ODE \( T'' = -\lambda c^2 T \)

\[ T(t) = A_n \cos \left( \frac{n\pi c}{L} t \right) + B_n \sin \left( \frac{n\pi c}{L} t \right) \]

Hence any function of the form

\[ u(x,t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{L} \right) \left( A_n \cos \left( \frac{n\pi c}{L} t \right) + B_n \sin \left( \frac{n\pi c}{L} t \right) \right) \]

satisfies the PDE and BCs.

Now choose \( A_n \) & \( B_n \) so that the two initial conditions are satisfied, i.e. \( \forall 0 \leq x \leq L, \)

\[ 2 \sin \left( \frac{3\pi x}{L} \right) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right) \]

"at rest" \( \sqrt{D} = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \left( \frac{n\pi x}{L} \right) \)

Hence \( B_1 = B_2 = \ldots = 0 \) & \( A_1 = A_2 = A_4 = \ldots = 0 \) while \( A_3 = \sqrt{D} \) Hence solution is

\[ u(x,t) = 2 \sin \left( \frac{3\pi x}{L} \right) \cos \left( \frac{3\pi c}{L} t \right) \]