FOURIER SERIES

We show that any reasonable function \( f: [-L, L] \rightarrow \mathbb{R} \) can be represented as a Fourier series

\[
j(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{\pi x}{L} + b_n \sin \frac{\pi x}{L} \right)
\]

where the coefficients \( a_n \) and \( b_n \) are defined appropriately.

The coefficients \( a_n \) and \( b_n \) can be found by considering orthogonal basis sets.

Vector space \( V \) equipped with

\[
\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}
\]

Inner product spaces - see handout for definition.

\[
\langle x_1, x_2 \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_N y_N
\]

\[ (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \]

Let \( V \) = family of all piecewise continuous functions from \([-L, L] \rightarrow \mathbb{R}\)

We can define an inner product on \( V \) by

\[
\langle f, g \rangle = \int_{-L}^{L} f(x)g(x) \, dx
\]

Orthogonal basis - see handout for definition

Lemma

Let \( \{ u_1, \ldots, u_N \} \) be an orthogonal family of non-zero vectors and suppose \( u = \sum_{i=1}^{N} c_i u_i \). Then \( c_i = \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} \).

Proof

Since \( u = c_1 u_1 + \ldots + c_N u_N \),

\[
\langle u, u_i \rangle = \langle c_1 u_1 + \ldots + c_N u_N, u_i \rangle = c_1 \langle u_1, u_i \rangle + \ldots + c_N \langle u_N, u_i \rangle = c_i \langle u_i, u_i \rangle \quad \text{as} \quad \langle u_i, u_i \rangle \neq 0 \quad \text{for} \quad i \neq 1.
\]

Hence \( c_i = \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} \).

Similarly \( c_i = \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} \).

Examples

(a) \( \mathbb{R}^2 \) has orthogonal basis \( \{ (1), (-1) \} \).

Suppose

\[
\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]


\[
u
\]

\[
u_1
\]

\[
u_2
\]
Let \( \mathcal{V} \) be function space and suppose \( \{ f_n \} \) is an orthogonal set in \( \mathcal{V} \).

Suppose \( u = c_1 f_1 + \ldots + c_N f_N \).

Then \( c_i = \frac{\langle u, f_i \rangle}{\langle f_i, f_i \rangle} = \frac{\int_a^L f_i(x) f_j(x) \, dx}{\int_a^L f_i(x)^2 \, dx} \).

**Full Range Fourier Series**

Suppose \( \mathcal{V} = \) family of all piecewise continuous functions on \([ -L, L ] \).

We now show that

\[
\{ \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \ldots, \sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \ldots \}
\]

is an orthogonal set in \( \mathcal{V} \).

Suppose \( m \) and \( n \) are positive integers.

\[
\int_{-L}^L \cos \frac{\pi x}{L} \cos \frac{m \pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases}
\]

Similarly, \( \int_{-L}^L \sin \frac{\pi x}{L} \cos \frac{m \pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases} \)

\[
\int_{-L}^L \sin \frac{\pi x}{L} \sin \frac{m \pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases}
\]

\[
\int_{-L}^L \cos \frac{\pi x}{L} \sin \frac{m \pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases}
\]

Similarly, \( \int_{-L}^L \cos \frac{m \pi x}{L} \cos \frac{\pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases} \)

\[
\int_{-L}^L \sin \frac{m \pi x}{L} \sin \frac{\pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases}
\]

\[
\int_{-L}^L \cos \frac{m \pi x}{L} \sin \frac{\pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{m} & \text{if } m = n \end{cases}
\]

Hence, \( \{ \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \ldots, \sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \ldots \} \) is an orthogonal family in \( \mathcal{V} \).
Thus if \( f \in C^1 \) and

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \frac{1}{n} \left( a_n \cos \frac{2\pi nt}{L} + b_n \sin \frac{2\pi nt}{L} \right),
\]

we must have

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx,
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{2\pi nx}{L} \, dx,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{2\pi nx}{L} \, dx.
\]

(3) is called the Fourier series of \( f \).

Roughly speaking, (3) holds when \( f \) is piecewise continuous.

**Definition**

A function is said to be piecewise continuous on \([a, b]\) if there exist a finite number of points \( a < x_1 < \cdots < x_n < b \) such that

- \( f \) is continuous on each subinterval \([x_{i-1}, x_i)\) and \( \lim_{x \to x_i^-} f(x) \) and \( \lim_{x \to x_i^+} f(x) \) exist for all \( i \)
- \( f \) is continuous except at \( x_1, \ldots, x_n \) where it has discontinuous jumps.

\[\text{graph}\]

**Remark**

The \( n \)-th of (3) has period \( 2L \).

Now make precise with relevant more precise:

**Theorem**

Let \( f \) and \( f' \) be piecewise continuous on the interval \(-L \leq x \leq L\) and suppose \( f \) is defined outside of \([-L, L]\) so that it is a periodic function of period \( 2L \). Then the Fourier series for \( f \) converges to \( f(x) \) at all points where \( f \) is continuous and to \( \frac{1}{2} \left[ f(x^+) + f(x^-) \right] \) where \( f \) is discontinuous.
Exercise

Find the Fourier series for \( f(x) = x \) \((-1 \leq x \leq 1)\)

Solution

Fourier series is

\[
    f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]

where

\[
    a_0 = \frac{1}{2} \int_{-1}^{1} x \, dx = 0 \quad (\text{as } x \text{ is an odd function so } f(-x) = -f(x))
\]

\[
    a_n = \frac{1}{\pi} \int_{-1}^{1} x \cos nx \, dx = 0 \quad (\text{as } x \cos nx \text{ is an odd function})
\]

\[
    b_n = \frac{1}{\pi} \int_{-1}^{1} x \sin nx \, dx = -\frac{1}{n^2 \pi} \cos nx \bigg|_{-1}^{1} = \frac{1}{n^2} \left[ -\cos n - (-1)^n \cos n \right]
\]

\[
    = -\frac{2}{n \pi} \cos n \pi \cdot (-1)^{n+1} \frac{2}{n^2}
\]

Hence we obtain the following Fourier series for \( f \):

\[
    f(x) \sim \frac{2}{\pi} \left( \sin nx - \frac{1}{2} \sin 2nx + \frac{1}{3} \sin 3nx - \frac{1}{4} \sin 4nx \ldots \right)
\]

Remark (a) To determine the sum of the Fourier series for all \( x \), we must extend \( f \) to be a periodic function of period 2 on all of \( \mathbb{R} \).

Then \( f \) is continuous for all \( x \in \mathbb{R} \) except for \( x = \pm 1, \pm 3 \) etc.

Hence

\[
    f(x) = \frac{2}{\pi} \left( \sin nx - \frac{1}{2} \sin 2nx + \frac{1}{3} \sin 3nx \ldots \right) \quad \text{for } x \in \{ \pm 1, \pm 3, \ldots \}
\]

(b) When \( x = \pm 1, \pm 3, \ldots \), clearly every term in Fourier series = 0; this is consistent with the fact that \( f(1^+) + f(1^-) = -1 + 1 = 0 \)

\( (c) \) Thus Fourier series converges to function with graph
The first $N$ terms of a Fourier series is

$$a_0 + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx$$

define a continuous function with graph.

\[ f(x) = \frac{1}{2} + \frac{3}{\pi} \cos \frac{x}{2} - \frac{3}{\pi} \sin \frac{x}{2} + \frac{3}{\pi} \cos \frac{x}{3} - \frac{1}{\pi} \sin \frac{x}{3} + \cdots \]

Hence

$$-\frac{3}{\pi} - \frac{1}{\pi} - \frac{3}{\pi} - \cdots = \frac{\pi}{4}.$$  

Often useful to know

\[ f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x \leq \pi \end{cases} \]

Find the Fourier series for

\[ f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x \leq \pi \end{cases} \]

Solution

Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 1 \, dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 1 \cdot \cos nx \, dx = \frac{1}{\pi} \sin nx \bigg|_{-\pi}^{0} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 1 \cdot \sin nx \, dx = \frac{1}{\pi} \cos nx \bigg|_{-\pi}^{0} = 0$$

$$= \frac{1}{\pi} \int \cos nx - 1 \, dx = \left\{ \begin{array}{ll} \frac{1}{n} \sin nx, & \text{if } n \text{ is even} \\ -\frac{2}{n} x, & \text{if } n \text{ is odd} \end{array} \right.$$}

Hence Fourier series for $f$ is

$$\frac{1}{2} - \frac{2}{\pi} \left[ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$$

Extending $f$ as a function with period $2\pi$ and remembering that the series converges to $\frac{1}{2} [f(x^+) + f(x^-)]$ where $f$ is discontinuous, we have that series converges to following limit.
At $x = \frac{m\pi}{L}$, series converges to $\frac{1}{2}$ (This is also obvious from series)  
Graph of sum of first $N$ terms of series is like

Half Range Fourier Series
We have shown above how to obtain a Fourier series containing both sine and cosine terms for a function $f$ on an interval $[-L, L]$
We now show how to obtain a series involving only sine terms (or only cosine terms) for $f$ on $[0, L]$
$[0, L]$ is half of $[-L, L]$ - hence the term half-range series

Suppose we are given $f : [0, L] \rightarrow \mathbb{R}$

Fourier Sine Series
Extend $f$ to be an odd function on $[-L, L]$, i.e., let $f(-x) = -f(x)$
Then on $[-L, L]$ $f$ has full range Fourier series

$$f(x) \sim a_0 + \frac{1}{L} \left( a_n \cos \frac{m\pi x}{L} + b_n \sin \frac{m\pi x}{L} \right)$$
But $a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = 0$ as $f$ is odd
and

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx = 0 \quad (as \ x \rightarrow 0 \ f(x) \cos \frac{m\pi x}{L} \text{ is odd})$$
Hence on $[-L, L]$ and so on $[0, L]$ we have

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{m\pi x}{L}$$
where

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{m\pi x}{L} \, dx$$
(As $x \rightarrow 0 \ f(x) \sin \frac{m\pi x}{L}$ is even).

Fourier Cosine Series
Extend $f$ to be an even function on $[-L, L]$ i.e. let $f(-x) = f(x)$
Then on $[-L, L]$ $f$ has full range Fourier series

$$f(x) \sim a_0 + \frac{1}{L} \sum_{n=1}^{\infty} a_n \cos \frac{m\pi x}{L} + b_n \sin \frac{m\pi x}{L}$$
Since $f$ is even we have

$$a_0 = \frac{1}{L} \int_0^L f(x) \, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{mn\pi}{L} \, dx$$

with $b_n = 0$.

Since $f$ is even, the Fourier cosine series is

$$f(x) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{mn\pi x}{L}$$

Exercise (i) Find the Fourier cosine series for $f(x) = x$ ($0 < x < 1$).

(ii) Sketch the graph of the function to which the series converges for $-3 < x < 3$.

(iii) Find $1 + \frac{1}{3} + \frac{1}{5} + \ldots$

Solution

Fourier cosine series is

$$f(x) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{mn\pi x}{L}$$

where

$$a_0 = \frac{1}{L} \int_0^L f(x) \, dx = \frac{1}{2}$$

and

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{mn\pi x}{L} \, dx = \frac{2}{mn\pi} \left[ \sin \frac{mn\pi}{L} \right]_0^L$$

$$= \frac{2}{mn\pi} \cos \frac{mn\pi}{L} \left[ \cos \frac{mn\pi}{L} - 1 \right]$$

$$= \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{2}{ mn\pi} & \text{if } n \text{ odd} \end{cases}$$

Hence we obtain cosine series

$$f(x) \approx \frac{1}{2} - \frac{4}{\pi^2} \left( \cos \pi x + \frac{1}{3} \cos 3\pi x + \frac{1}{5} \cos 5\pi x + \ldots \right)$$

(iv) Extending $f$ to an even function with period $2L$ we have

Since this function is continuous, the Fourier series converges to $f(x)$ for all $x$.
(iii) Setting \( x = 0 \) we obtain

\[ f(0) = 0 = \frac{a_0}{2} - \frac{4}{\pi^2} \left(1 + \frac{1}{4} + \frac{1}{16} + \ldots\right) \]

Hence \( 1 + \frac{1}{4} + \frac{1}{16} + \ldots = \frac{\pi^2}{8} \).

(ii) Repeat (i) and (ii) for Fourier sine series.

The D.T.

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{\pi nx}{\ell}\right) \quad b_n = 2 \int_0^\ell x \sin \left(\frac{\pi nx}{\ell}\right) \, dx \]

\[ = -2 \left[ \frac{x \cos \left(\frac{\pi nx}{\ell}\right)}{\frac{\pi n}{\ell}} \right]_0^\ell + \frac{1}{\frac{\pi n}{\ell}} \left[ \sin \left(\frac{\pi nx}{\ell}\right) \right]_0^\ell \]

\[ = \frac{2(-1)^{n+1}}{n\ell} \]

So \( F_S \rightarrow \frac{2}{\ell} \left( \frac{\sin \left(\frac{\pi x}{\ell}\right)}{3} - \frac{\sin \left(\frac{2\pi x}{\ell}\right)}{3} + \frac{\sin \left(3\pi x/\ell\right)}{3} \right) \).

(iii) Converges to