LAPLACE'S EQUATION

The steady state temperature in a body $D \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ satisfies

$$\nabla^2 u(x) = 0 \text{ for } x \in D$$

$u$ satisfies given boundary conditions on $\partial D$.

This is known as Laplace's equation - the equation also arises in electromagnetism, fluid flow and elasticity.

We shall show how to solve Laplace's equation in rectangles and circles.

(a) Laplace's equation in a rectangle

Let $R = [0, a] \times [0, b]$.

Firstly, we shall find a solution of

$$u_{xx} + u_{yy} = 0 \text{ for } (x, y) \in R$$

$$u(x, 0) = f(x) \text{ for } 0 \leq x \leq a; \quad u(x, b) = 0 \text{ for } 0 \leq x \leq a$$

$$u(0, y) = 0 \text{ for } 0 \leq y \leq b; \quad u(a, y) = 0 \text{ for } 0 \leq y \leq b$$

We seek a solution of the form $u(x, y) = X(x)Y(y)$.

To ensure that $u(x, y)$ satisfies the zero boundary conditions at $x = 0$, $x = a$, $y = 0$, $y = b$, we require $X(0) = 0 = X(a)$, $Y(b) = 0$.

Also $u$ is a solution if

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

ie.

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

for some constant $\lambda$.

ie.

$$-X''(x) = \lambda X(x) \quad X(0) = 0 = X(a)$$

(1)

$$-Y''(y) = \lambda Y(y) \quad Y(b) = 0$$

(2)

(1) has non-zero solutions if $\lambda = \frac{n^2 \pi^2}{a^2}$ where $n = 1, 2, \ldots$ and the corresponding eigenfunctions are $\sin \frac{n \pi x}{a}$.

If $\lambda = \frac{n^2 \pi^2}{a^2}$, (1) has solution $A \sin \frac{n \pi x}{a}$.
Hence (x) has a solution \( \phi(x, y) = \sum_{m=1}^{\infty} A_m \sinh \frac{m \pi x}{a} \sinh \frac{m \pi y}{b} \) for \( n = 1, 2 \), which satisfies the zero boundary conditions.

Since (x) is linear, we also have the solution

\[ \phi(x, y) = \sum_{m=1}^{\infty} A_m \sinh \frac{m \pi x}{a} \sinh \frac{m \pi y}{b} \]

and must choose the \( A_m \)'s so that the boundary condition \( \phi(x, 0) = f(x) \) is satisfied.

Thus we must choose the \( A_m \)'s so that

\[
\phi(x, y) = \frac{1}{a} \sum_{m=1}^{\infty} A_m \int_{0}^{a} f(x) \sinh \frac{m \pi x}{a} \, dx \]

Thus we have obtained the solution

\[
\phi(x, y) = \frac{1}{a} \sum_{m=1}^{\infty} \left( \int_{0}^{a} f(x) \sinh \frac{m \pi x}{a} \, dx \right) \sinh \frac{m \pi y}{b} \sinh \frac{m \pi x}{a}
\]

Remarks (i): Similar arguments can be used to solve

\[ \nabla^2 \phi(x, y) = 0 \quad \text{for} \quad (x, y) \in \mathbb{R} \]

together with one of the boundary conditions.

(iii) If we have boundary condition

\[
\phi(x, 0) = \begin{cases} f(x) & \text{if } x < 0 \\ f(y) & \text{if } y < 0 \\ f(x) & \text{if } y > 0 \end{cases}
\]

we proceed by solving \( \nabla^2 \phi = 0 \) subject to each of the following boundary conditions.
If the solutions are \( u_1, u_2, u_3, u_4 \) then \( u - u_1 + u_2 + u_3 + u_4 \) is a solution of the original problem.

(b) Laplace's equation on a circle.

We shall solve

\[
\begin{align*}
\Delta^2 u(x, y) &= 0 \quad \text{for } x^2 + y^2 < a^2 \\
u(x, y) &= f(x, y) \quad \text{for } x^2 + y^2 = a^2
\end{align*}
\]

In order to make separation of variables possible it is necessary to use polar co-ordinates.

Let \( x = r \cos \theta \) and \( y = r \sin \theta \). By using the chain rule the equation can be expressed as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < a, \quad 0 \leq \theta \leq 2\pi
\]

\[
u(a, \theta) = f(\theta) \quad \text{for } 0 \leq \theta \leq 2\pi.
\]

We seek solutions of the form \( u(r, \theta) = R(r) \Theta(\theta) \).

Clearly \( \Theta \) must have period \( 2\pi \).

Also we must have

\[
R'' + \frac{1}{r} R' + \frac{1}{r^2} R = 0
\]

and so

\[
r^2 R'' + r R' - k^2 R = 0
\]

\[
-\Theta'' + k^2 \Theta = 0
\]

where \( k \) is a constant.

(2) has periodic solutions only when \( k > 0 \) and in this case (2) has general solution \( A_\Theta \cos k \theta + B_\Theta \sin k \theta \).

Hence (2) has periodic solutions of period \( 2\pi \) if and only if

\[
\sqrt{k} = m \left( \frac{\pi}{a} \right), \quad m = 0, 1, 2, \ldots
\]

(3) has solutions only when \( \sqrt{k} > 0 \) and in this case (3) has general solution \( A_\Phi \cos \sqrt{k} x + B_\Phi \sin \sqrt{k} x \).

Hence (3) has periodic solutions of period \( 2\pi \) if and only if

\[
\sqrt{k} = m \left( \frac{\pi}{a} \right), \quad m = 0, 1, 2, \ldots
\]
When \( n = 0 \) we have the constant solution

When \( n \geq 2 \), \( n = 1, 2 \ldots \) we have solution \( A_n \cos n\theta + B_n \sin n\theta \)

(1) is an Euler type equation which can be solved using the change

variable \( u = e^t \) so that

\[
\frac{d^2 u}{dt^2} = \frac{1}{e^t} \frac{d^2 u}{dt^2} + \frac{d^2 u}{dt^2} = \frac{1}{e^t} \frac{d^2 u}{dt^2} - \frac{1}{e^t} \frac{d^2 u}{dt^2}
\]

If \( b = 0 \), (1) becomes \( \frac{d^2 R}{dt^2} = 0 \) \( \Rightarrow R(t) = \frac{1}{2} t^2 + \frac{1}{2} bnt \)

If \( b = n^2 \), (1) becomes \( \frac{d^2 R}{dt^2} - n^2 R = 0 \) \( \Rightarrow R(t) = e^{nt}, e^{-nt} \)

As the equation must be satisfied at all points on the circle we require \( R(x) \) bounded and so we shall use only the solutions 1 and \( e^{nt} \)

Thus we have obtained solutions \( 1 \) and \( (A_n \cos n\theta + B_n \sin n\theta)^\gamma \)

We now seek a solution of the form

\[
u(x, \theta) = A_0 + \sum_{n=1}^\infty (A_n \cos n\theta + B_n \sin n\theta) x^n \]

which satisfies the boundary condition \( \nu(a, \theta) = f(\theta) \).

Thus we require

\[
f(x) = A_0 + \sum_{n=1}^\infty (A_n \cos n\theta + B_n \sin n\theta) a^n \quad \text{Fourier series}
\]

and so must choose \( A_0, A_n, B_n \) so that \( A_0, a^n A_n, a^n B_n \) are.

The coefficients in the full range Fourier series of \( f(x) \).

\[
A_n = \frac{1}{a^n} \int_{-\pi}^{\pi} f(x) \cos n\theta \, dx, \quad B_n = \frac{1}{a^n} \int_{-\pi}^{\pi} f(x) \sin n\theta \, dx
\]

We can adapt the above argument to deal with other circle-like regions

Suppose \( V^2 u = 0 \) is satisfied on

(1) on a semi-circle with the following boundary conditions

Then if \( \nabla \theta = \frac{\partial}{\partial \theta} \) \( \theta(\theta) \) equations (1) and (2) become
\[ x^2 R'' + 2xR' - \lambda^2 R = 0 \quad R(1) = 0 \quad (1) \]
\[ -\Theta'' = \lambda^2 \Theta, \quad \Theta(0) = \Theta(\pi) = 0 \quad (2) \]

(2) has non-zero solutions when \( \lambda = n^2 \), \( n = 1, 2, \ldots \) with corresponding eigenfunctions \( \sin n\Theta \).

Thus we have solution
\[ u(r, \Theta) = \sum_{n=1}^{\infty} A_n \sin n\Theta \cdot r^n \]
and we require
\[ f(\Theta) = \frac{1}{\pi} \int_{0}^{\pi} f(\Theta) \sin n\Theta \, d\Theta \]

on the quarter circle
\[ \text{(ii)} \]

We obtain
\[ x^2 R'' + xR' - \lambda^2 R = 0 \quad R(1) = 0 \quad (1) \]
\[ -\Theta'' = \lambda^2 \Theta, \quad \Theta(0) = \Theta(\pi) = 0 \quad (2) \]

(2) has non-zero solutions when \( \lambda = n^2 \), \( n = 1, 2, \ldots \) with corresponding eigenfunctions \( \sin n\Theta \).

Thus we have solution
\[ u(r, \Theta) = \sum_{n=1}^{\infty} A_n \sin n\Theta \cdot r^n \]
and we require
\[ f(\Theta) = \frac{1}{\pi} \int_{0}^{\pi} f(\Theta) \sin n\Theta \, d\Theta \]

on the annulus \( 1 < r < \infty \)

We obtain
\[ x^2 R'' + xR' - \lambda^2 R = 0 \quad R(1) = 0 \quad (1) \]
\[ -\Theta'' = \lambda^2 \Theta, \quad \Theta(0) = \Theta(\pi) = 0 \quad (2) \]

(2) has non-zero solutions when \( \lambda = n^2 \), \( n = 1, 2, \ldots \) with corresponding solutions \( \Theta(n\Theta) \), \( \cos n\Theta \), \( \sin n\Theta \), \( \ldots \).
When \( k = 0 \), (2) has solution 1. Also (1) has general solution
\[ A + B \ln r \] and so we have solution \( \ln r \) satisfying \( R(1) = 0 \)

When \( k = m^2 \), (2) has general solution \( A_n \cos \theta + B_n \sin \theta \)
Also (1) has general solution \( A \cos m \theta + B \sin m \theta \) and so we have solution
\[ r^{-m} - r^m \] satisfying \( R(1) = 0 \)

Thus we have solution
\[ u(x, \theta) = A_0 \ln r + \sum_{m=1}^{\infty} \frac{(A_m \cos m \theta + B_m \sin m \theta)}{m} \left( r^m - r^{-m} \right) \]

and we require
\[ u(x, \theta) = f(\theta) = A_0 \ln r + \sum_{m=1}^{\infty} \frac{(A_m \cos m \theta + B_m \sin m \theta)}{m} \left( r^m - r^{-m} \right) \]

and we can find \( A_0, A_m, B_m \).

Uniqueness of Solutions of Laplace's Equation

We have shown how to find a solution of
\[ \nabla^2 u(x) = 0 \quad \text{for } x \in D \] (x)

\[ u(x) = f(\theta) \quad \text{for } x \in \partial D \]

for certain regions \( D \).

We now show that the solution we have found is the only possible solution.

We require the following 'integration by parts' result.

Lemma
Let \( u, v : \mathbb{R}^n \to \mathbb{R} \) be smooth functions and \( D \subset \mathbb{R}^n \) be a bounded region with smooth boundary. Then
\[ \int_D \nabla^2 u(x) \nabla v(x) \, dx = -\int_{\partial D} u(x) \frac{\partial v}{\partial n} \, ds \]

Remarks

(i) we integrate \( \nabla^2 u \) and differentiate \( u \)

(ii) the \( n = 1 \) version is
\[ \int_0^a u_{xx}(x) \, dx \, dx \, dx = u_x(x) \left|_{x=1}^{x=0} \right. \]

Let \( u \) be a solution of (x).

We can now prove uniqueness.

Suppose \( u \) and \( v \) are both solutions of (x).

Let \( w = u - v \). It is easy to see that \( w \) satisfies
\( \nabla^2 w(x) = 0 \quad \text{for} \quad x \in D \)
\[ w(x) = 0 \quad \text{for} \quad x \in \partial D \]

Hence
\[ 0 = \int_D \nabla^2 w(x) \, dx = \int_D \frac{\partial}{\partial x} \left( \nabla w(x) \cdot n(x) \right) \, dx \]
\[ - \int_D \left| \nabla w(x) \right|^2 \, dx \]
\[ = - \int_D \nu \cdot \nu \, dx \quad (\text{as} \quad w(x) \equiv 0 \quad \text{for} \quad x \in \partial D) \]

Thus \( \int_D \left| \nabla w(x) \right|^2 \, dx = 0 \) and so \( w(x) \equiv 0 \) for \( x \in D \).

Hence \( u \) must be a constant function on \( D \) and so, since \( w(x) \equiv 0 \) for \( x \in \partial D \), \( \nu(x) \equiv 0 \) for \( x \in D \).

Thus \( u = c \) and so \( (w) \) has at most one solution.