An introduction to asymptotic analysis

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Contents

CHAPTER 1

Order notation

The symbols $\mathcal{O}$, o and $\sim$, were first used by E. Landau and P. Du Bois-Reymond and are defined as follows. Suppose $f(z)$ and $g(z)$ are functions of the continuous complex variable z defined on some domain $\mathcal{D} \subset \mathbb{C}$ and possess limits as $z \to z_0$ in $\mathcal{D}$. Then we define the following shorthand notation for the *relative properties* of these functions in the limit $z \to z_0$.

Asymptotically bounded:

$$f(z) = \mathcal{O}(g(z)) \quad \text{as} \quad z \to z_0\,,$$

means that: there exists constants $K \geq 0$ and $\delta > 0$ such that, for $0 < |z - z_0| < \delta$,

$$|f(z)| \leq K|g(z)|\,.$$

We say that $f(z)$ is *asymptotically bounded* by $g(z)$ in magnitude as $z \to z_0$, or more colloquially, and we say that $f(z)$ is of 'order big O' of $g(z)$. Hence provided $g(z)$ is not zero in a neighbourhood of z_0, except possibly at z_0, then

$$\left|\frac{f(z)}{g(z)}\right| \text{ is bounded}\,.$$

Asymptotically smaller:

$$f(z) = \mathrm{o}(g(z)) \quad \text{as} \quad z \to z_0\,,$$

means that: for all $\epsilon > 0$, there exists $\delta > 0$ such that, for $0 < |z - z_0| < \delta$,

$$|f(z)| \leq \epsilon|g(z)|\,.$$

Equivalently this means that, provided $g(z)$ is not zero in a neighbourhood of z_0 except possibly at z_0, then as $z \to z_0$:

$$\frac{f(z)}{g(z)} \to 0\,.$$

We say that $f(z)$ is *asymptotically smaller* than $g(z)$, or more colloquially, $f(z)$ is of 'order little o' of $g(z)$, as $z \to z_0$.

Asymptotically equal:

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0 \,,$$

means that, provided $g(z)$ is not zero in a neighbourhood of z_0 except possibly at z_0, then as $z \to z_0$:

$$\frac{f(z)}{g(z)} \to 1 \,.$$

Equivalently this means that as $z \to z_0$:

$$f(z) = g(z) + \mathrm{o}(g(z)) \,.$$

We say that $f(z)$ *asymptotically equivalent* to $g(z)$ in this limit, or more colloquially, $f(z)$ 'goes like' $g(z)$ as $z \to z_0$.

Note that $\mathcal{O}$-order is more informative than o-order about the behaviour of the function concerned as $z \to z_0$. For example, $\sin z = z + \mathrm{o}(z^2)$ as $z \to 0$ tells us that $\sin z - z \to 0$ faster than z^2, however $\sin z = z + \mathcal{O}(z^3)$, tells us specifically that $\sin z - z \to 0$ like z^3.

Examples.

- $f(t) = \mathcal{O}(1)$ as $t \to t_0$ means $f(t)$ is bounded when t is close to t_0.
- $f(t) = \mathrm{o}(1) \Rightarrow f(t) \to 0$ as $t \to t_0$.
- If $f(t) = 5t^2 + t + 3$, then $f(t) = o(t^3)$, $f(t) = \mathcal{O}(t^2)$ and $f(t) \sim 5t^2$ as $t \to \infty$; but $f(t) \sim 3$ as $t \to 0$ and $f(t) = \mathrm{o}(1/t)$ as $t \to \infty$.
- As $t \to \infty$, $t^{1000} = o(e^t)$, $\cos t = \mathcal{O}(1)$.
- As $t \to 0+$, $t^2 = o(t)$, $e^{-1/t} = o(1)$, $\tan t = \mathcal{O}(t)$, $\sin t \sim t$.
- As $t \to 0$, $\sin(1/t) = \mathcal{O}(1)$, $\cos t \sim 1 - \frac{1}{2}t^2$.

Remarks.

(1) In the definitions above, the function $g(z)$ is often called a *gauge function* because it is the function against which the behaviour of $f(z)$ is gauged.

(2) This notation is also easily adaptable to functions of a discrete variable such as sequences of real numbers (i.e. functions of the positive integer n). For example, if $x_n = 3n^2 - 7n + 8$, then $x_n = o(n^3)$, $x_n = \mathcal{O}(n^2)$ and $x_n \sim 3n^2$ as $n \to \infty$.

(3) Often the alternative notation $f(z) \ll g(z)$ as $z \to z_0$ is used in place of $f(z) = \mathrm{o}(g(z))$ as $z \to z_0$.

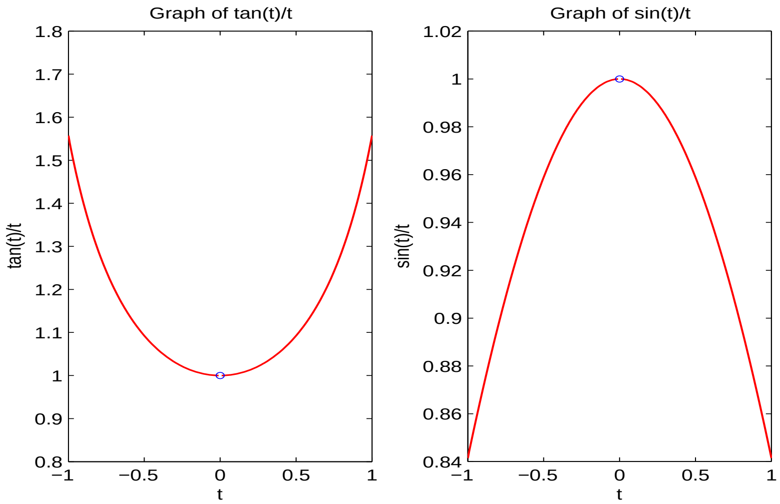


FIGURE 1. The behaviour of the functions $\tan(t)/t$ and $\sin(t)/t$ near $t = 0$. These functions are undefined at $t = 0$; but both these functions approach the value 1 as t approaches 0 from the left and the right.

CHAPTER 2

Perturbation methods

Usually in applied mathematics, though we can write down the equations for a model, we cannot always solve them, i.e. we cannot find an analytical solution in terms of known functions or tables. However an approximate answer may be sufficient for our needs provided we know the size of the error and we are willing to accept it. Typically the first recourse is to numerically solve the system of equations on a computer to get an idea of how the solution behaves and responds to changes in parameter values. However it is often desirable to back-up our numerics with approximate analytical answers. This invariably involves the use of perturbation methods which try to exploit the smallness of an inherent parameter. Our model equations could be a system of algebraic and/or differential and/or integral equations, however here we will focus on scalar algebraic equations as a simple natural setting to introduce the ideas and techniques we need to develop (see Hinch [**5**] for more details).

2.1. Regular perturbation problems

Example. Consider the following quadratic equation for x which involves the small parameter ϵ:

$$x^2 + \epsilon x - 1 = 0\,, \tag{2.1}$$

where $0 < \epsilon \ll 1$. Of course, in this simple case we can solve the equation exactly so that

$$x = -\tfrac{1}{2}\epsilon \pm \sqrt{1 + \tfrac{1}{4}\epsilon^2}\,,$$

and we can expand these two solutions about $\epsilon = 0$ to obtain the binomial series expansions

$$x = \begin{cases} +1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + \mathcal{O}(\epsilon^6)\,, \\ -1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{128}\epsilon^4 + \mathcal{O}(\epsilon^6)\,. \end{cases} \tag{2.2}$$

Though these expansions converge for $|\epsilon| < 2$, a more important property is that low order truncations of these series are good approximations to the roots when ϵ is small, and maybe more efficiently evaluated (in terms of computer time) than the exact solution which involves square roots.

However for general equations we may not be able to solve for the solution exactly and we will need to somehow derive analytical approximations when ϵ is small, from scratch. There are two main methods and we will explore the techniques involved in both of them in terms of our simple example (2.1).

Expansion method. The idea behind this method is to formally expand the solution about one of the unperturbed roots, say $x_0 = +1$, as a power series in ϵ:

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots ,$$

where the coefficients, x_1, x_2, x_3, ... are a-priori unknown. We then substitute this expansion into the quadratic equation (2.1) and formally equate powers of ϵ (assuming that such manipulations are permissible):

$$\begin{aligned}(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots)^2 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots) - 1 = 0\\ \Leftrightarrow \quad \big(1 + \epsilon(2x_1) + \epsilon^2(2x_2 + x_1^2) + \epsilon^3(2x_3 + 2x_1x_2) + \cdots\big)\\ + \big(\epsilon + \epsilon^2 x_1 + \epsilon^3 x_2 + \cdots\big) - 1 = 0\,.\end{aligned}$$

Now equating the powers of ϵ on both sides of the equation

$$\begin{aligned}\epsilon^0 &: \quad 1 - 1 = 0\,,\\ \epsilon^1 &: \quad 2x_1 + 1 = 0 \quad \Rightarrow \quad x_1 = \tfrac{1}{2}\,,\\ \epsilon^2 &: \quad 2x_2 + x_1^2 + x_1 = 0 \quad \Rightarrow \quad x_2 = \tfrac{1}{8}\,,\\ \epsilon^3 &: \quad 2x_3 + 2x_1x_2 + x_2 = 0 \quad \Rightarrow \quad x_3 = 0\,,\end{aligned}$$

and so on. Note that the first equation is trivial since we actually expanded about the $\epsilon = 0$ solution, namely $x_0 = +1$. Hence we see that

$$x(\epsilon) = 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^4)\,.$$

For ϵ small, this expansion truncated after the third term is a good approximation to the actual positive root of (2.1). We would say it is an *order 4 approximation* as the error we incur due to truncation is a term of $\mathcal{O}(\epsilon^4)$. We can obtain approximate solution expansions another way, via the so-called iterative method which we investigate next.

Iteration method. When $\epsilon = 0$ the quadratic equation (2.1) reduces to $x^2 - 1 = 0 \Leftrightarrow x = \pm 1$. For ϵ small, we expect the roots of the full quadratic equation (2.1) to be close to ± 1. Let's focus on the positive root; naturally we should take $x_0 = +1$ as our initial guess for this root for ϵ small.

The first step in the iterative method is to find a suitable rearrangement of the original equation that will be a suitable basis for an iterative scheme. Recall that equations of the form

$$x = f(x) \tag{2.3}$$

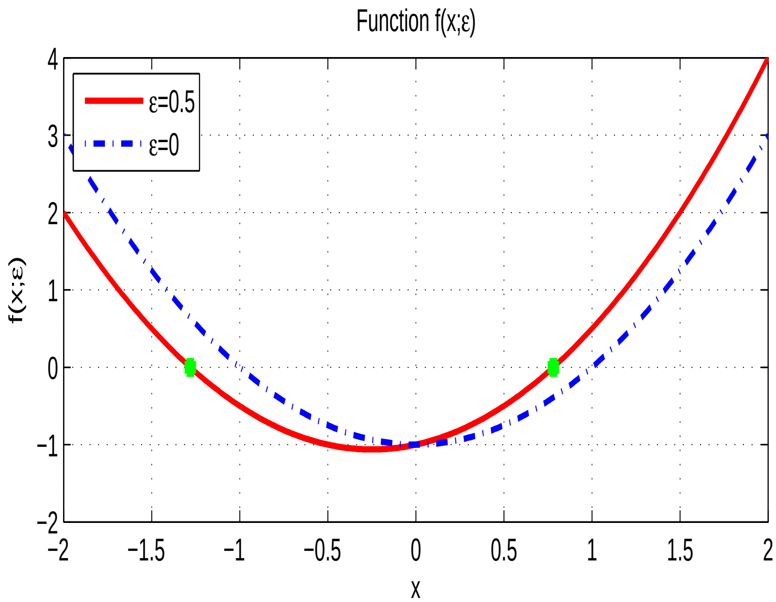
can be solved by using the iteration scheme (for $n \geq 0$):

$$x_{n+1} = f(x_n)\,,$$

for some sufficiently accurate initial guess x_0. Such an iteration scheme *will* converge to the root of equation (2.3) provided that $|f'(x)| < 1$ for all x close to the root. There are many ways to rearrange equation (2.1) into the form (2.3); a suitable one is

$$x = \pm\sqrt{1 - \epsilon x}\,.$$

Note that the solutions of this rearrangement coincide with the solutions of (2.1). Since we are interested in the root close to 1, we will only consider



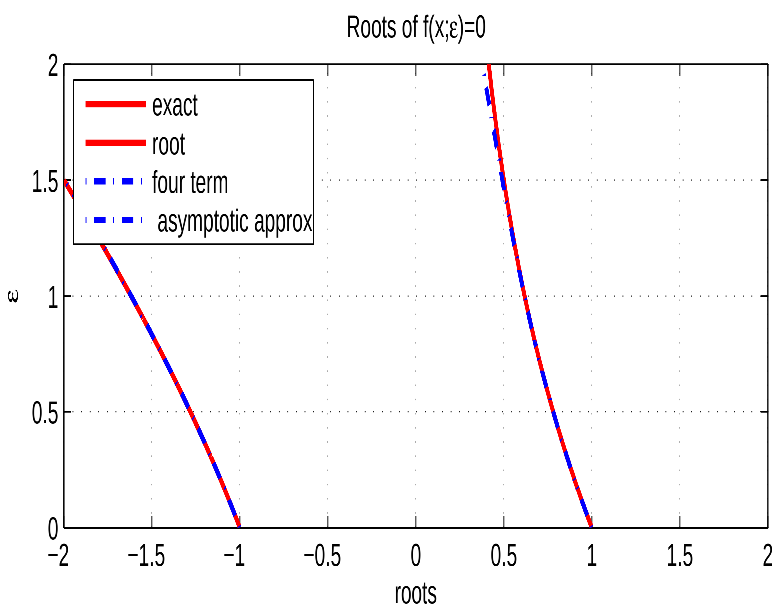


FIGURE 1. In the top figure we see how the quadratic function $f(x;\epsilon) = x^2 + \epsilon x - 1$ behaves while below we see how its roots evolve, as ϵ is increased from 0. The dotted curves in the lower figure are the asymptotic approximations for the roots.

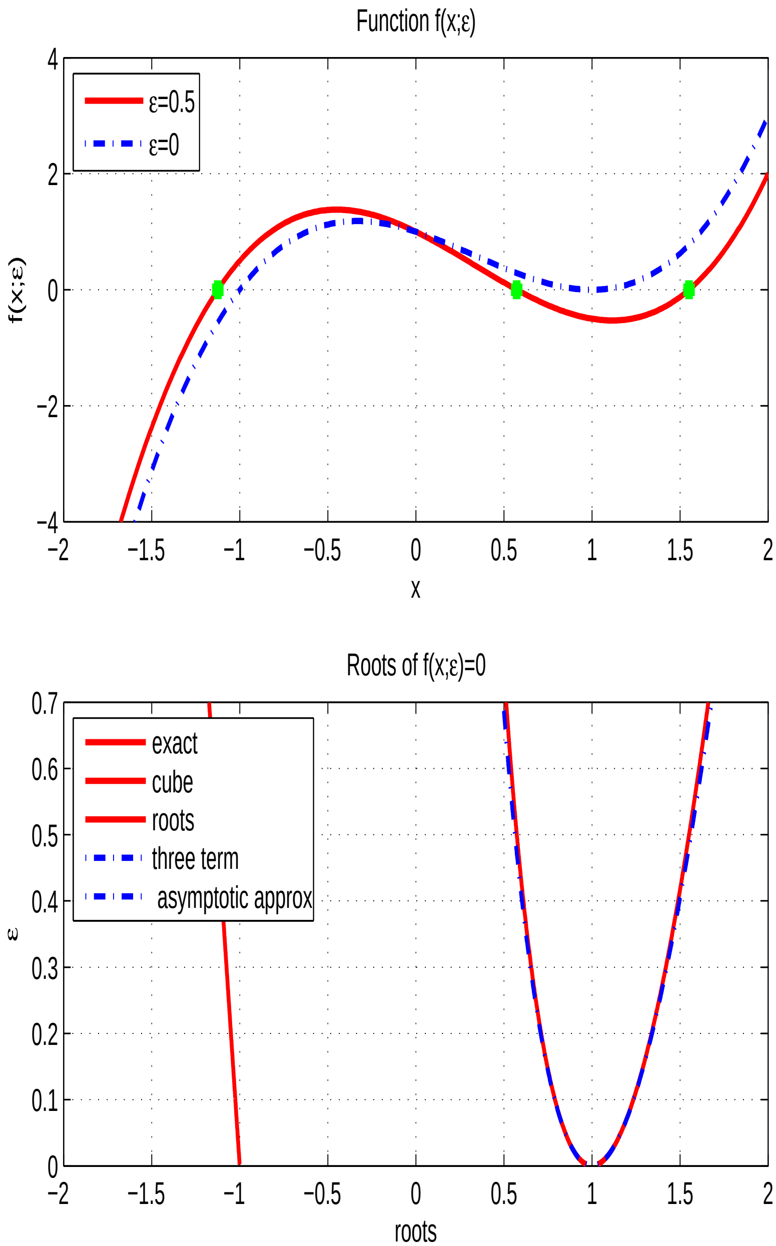


FIGURE 2. In the top figure we see how the cubic function $f(x;\epsilon) = x^3 - x^2 - (1+\epsilon)x + 1$ behaves while below we see how its roots evolve, as ϵ is increased from 0. The dotted curves in the lower figure are the asymptotic approximations for the roots close to 1.

the positive square root (we would take the negative square root if we were interested in approximating the root close to -1). Hence we identify $f(x) \equiv +\sqrt{1-\epsilon x}$ in this case and we have a rearrangement of the form (2.3). Also note that this is a sensible rearrangement as

$$\begin{aligned}
\left|f'(x)\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-\epsilon x}\right)\right| \\
&= \left|\frac{-\epsilon/2}{\sqrt{1-\epsilon x}}\right| \\
&\approx |-\epsilon/2|\,.
\end{aligned}$$

In the last step we used that $(1-\epsilon x)^{-1/2} \approx 1$ since x is near 1 and ϵ is small. In other words we see that close to the root

$$\left|f'(x)\right| \approx \epsilon/2\,,$$

which is small when ϵ is small. Hence the iteration scheme

$$x_{n+1} = \sqrt{1-\epsilon x_n}\,. \tag{2.4}$$

will converge. We take $x_0 = 1$ as our initial guess for the root.

Computing the first approximation using (2.4), we get

$$\begin{aligned}
x_1 =& \sqrt{1-\epsilon} \\
=& 1 - \tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 - \tfrac{1}{16}\epsilon^3 - \cdots\,,
\end{aligned}$$

where in the last step we used the binomial expansion. Comparing this with the expansion of the actual solution (2.2) we see that the terms of order ϵ^2 are incorrect. To proceed we thus truncate the series after the second term so that $x_1 = 1 - \frac{1}{2}\epsilon$ and iterate again:

$$\begin{aligned}
x_2 &= \sqrt{1-\epsilon\left(1-\tfrac{1}{2}\epsilon\right)} \\
&= 1 - \frac{1}{2}\epsilon\left(1-\tfrac{1}{2}\epsilon\right) - \tfrac{1}{8}\epsilon^2\left(1+\cdots\right)^2 + \cdots \\
&= 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + \cdots\,.
\end{aligned}$$

The term of order ϵ^2 is now correct and we truncate x_2 just after that term and iterate again:

$$\begin{aligned}
x_3 &= \sqrt{1-\epsilon\left(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2\right)} \\
&= 1 - \tfrac{1}{2}\epsilon\left(1-\tfrac{1}{2}\epsilon+\tfrac{1}{8}\epsilon^2\right) - \tfrac{1}{8}\epsilon^2\left(1-\tfrac{1}{2}\epsilon+\cdots\right)^2 - \tfrac{1}{16}\epsilon^3\left(1+\cdots\right)^3 + \cdots \\
&= 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + 0\cdot\epsilon^3 + \cdots\,.
\end{aligned}$$

We begin to see that as we continue the iterative process, more work is required—and to ensure that the current highest term is correct we need to take a further iterate.

Remark. Note that the size of $|f'(x)|$ for x near the root indicates the order of improvement to be expected from each iteration.

Example (non-integral powers). Find the first four terms in power series approximations for the root(s) of

$$x^3 - x^2 - (1+\epsilon)x + 1 = 0\,, \tag{2.5}$$

near $x = 1$, where ϵ is a small parameter. Let's proceed as before using the trial expansion method. First we note that at leading order (when $\epsilon = 0$) we have that

$$x_0^3 - x_0^2 - x_0 + 1 = 0\,,$$

which by direct substitution, clearly has a root $x_0 = 1$. If we assume the trial expansion

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\,,$$

and substitute this into (2.5), we soon run into problems when trying to determine x_1, x_2, etc... by equating powers of ϵ—try it and see what happens!

However, if we go back and examine the equation (2.5) more carefully, we realize that the root $x_0 = 1$ is rather special, in fact it is a double root since

$$x_0^3 - x_0^2 - x_0 + 1 = (x_0 - 1)^2 (x_0 + 1)\,.$$

(The third and final root $x_0 = -1$ is a single ordinary root.) Whenever we see a double root, this should give us a warning that we should tread more carefully.

Since the cubic $x^3 - x^2 - (1+\epsilon)x + 1$ has a double root at $x = 1$ when $\epsilon = 0$, this means that it behaves locally quadratically near $x = 1$. Hence an order ϵ change in x from $x = 1$ will produce and order ϵ^2 change in the cubic function (locally near $x = 1$). Equivalently, an order $\epsilon^{\frac{1}{2}}$ change in x locally near $x = 1$, will produce an order ϵ change in the cubic polynomial. This suggests that we instead should try a trial expansion in powers of $\epsilon^{\frac{1}{2}}$:

$$x(\epsilon) = x_0 + \epsilon^{\frac{1}{2}} x_1 + \epsilon x_2 + \epsilon^{\frac{3}{2}} x_3 + \cdots\,,$$

Substituting this into the cubic polynomial (2.5) we see that

$$\begin{aligned}
0 &= x^3 - x^2 - (1+\epsilon)x + 1 \\
&= \Big(1 + \epsilon^{\frac{1}{2}} x_1 + \epsilon x_2 + \epsilon^{\frac{3}{2}} x_3 + \cdots\Big)^3 - \Big(1 + \epsilon^{\frac{1}{2}} x_1 + \epsilon x_2 + \epsilon^{\frac{3}{2}} x_3 + \cdots\Big)^2 \\
&\quad - (1+\epsilon)\Big(1 + \epsilon^{\frac{1}{2}} x_1 + \epsilon x_2 + \epsilon^{\frac{3}{2}} x_3 + \cdots\Big) + 1 \\
&= \Big(1 + 3\epsilon^{\frac{1}{2}} x_1 + \epsilon(3x_1^2 + 3x_2) + \epsilon^{\frac{3}{2}}(x_1^3 + 6x_1x_2 + 3x_3) + \cdots\Big) \\
&\quad + \Big(1 + 2\epsilon^{\frac{1}{2}} x_1 + \epsilon(x_1^2 + 2x_2) + \epsilon^{\frac{3}{2}}(2x_1x_2 + 2x_3) + \cdots\Big) \\
&\quad - \Big(1 + \epsilon^{\frac{1}{2}} x_1 + \epsilon(1 + x_2) + \epsilon^{\frac{3}{2}}(x_1 + x_3) + \cdots\Big) + 1\,.
\end{aligned}$$

Hence equating coefficients of powers of ϵ:

$$\begin{aligned}
\epsilon^0 &: \quad 1 - 1 - 1 + 1 = 0 \\
\epsilon^{\frac{1}{2}} &: \quad 3x_1 - 2x_1 - x_1 = 0 \\
\epsilon &: \quad 3x_1^2 + 3x_2 - x_1^2 - 2x_2 - 1 - x_2 = 0 \quad \Rightarrow \quad x_1 = \pm\tfrac{1}{\sqrt{2}} \\
\epsilon^{\frac{3}{2}} &: \quad x_1^3 + 6x_1x_2 + 3x_3 - 2x_1x_2 - 2x_3 - x_1 - x_3 = 0 \quad \Rightarrow \quad x_2 = \tfrac{1}{8}\,.
\end{aligned}$$

Hence

$$x(\epsilon) \sim 1 \pm \tfrac{1}{\sqrt{2}}\epsilon^{\frac{1}{2}} + \tfrac{1}{8}\epsilon + \cdots .$$

Remark. If we were to try the iterative method, we might try to exploit the significance of $x = 1$, and choose the following decomposition for an appropriate iterative scheme:

$$(x-1)^2(x+1) = \epsilon x \quad \Rightarrow \quad x = 1 \pm \sqrt{\frac{\epsilon x}{1+x}} .$$

Example (non-integral powers). Find the first three terms in power series approximations for the root(s) of

$$(1-\epsilon)x^2 - 2x + 1 = 0 , \tag{2.6}$$

near $x = 1$, where ϵ is a small parameter.

Remark. Once we have realized that we need to pose a formal power expansion in say powers of $\epsilon^{\frac{1}{n}}$, we could equivalently set $\delta = \epsilon^{\frac{1}{n}}$ and expand in integer powers of δ. At the very end we simply substitute back that $\delta = \epsilon^{\frac{1}{n}}$. This approach is particularly convenient when you use either Maple[1] to try to solve such problems perturbatively.

Example (transcendental equation). Find the first three terms in the power series approximation of the root of

$$\mathrm{e}^x = 1 + \epsilon , \tag{2.7}$$

where ϵ is a small parameter.

2.2. Singular perturbation problems

Example. Consider the following quadratic equation:

$$\epsilon x^2 + x - 1 = 0 . \tag{2.8}$$

The key term in this last equation that characterizes the number of solutions is the first quadratic term ϵx^2. This term is 'knocked out' when $\epsilon = 0$. In particular we notice that when $\epsilon = 0$ there is only one root to the equation, namely $x = 1$, whereas for $\epsilon \neq 0$ there are two! Such cases, where the character of the problem changes significantly from the case when $0 < \epsilon \ll 1$ to the case when $\epsilon = 0$, we call *singular perturbation problems*. Problems that are not singular are *regular*.

For the moment, consider the exact solutions to (2.8) which can be determined using the quadratic equation formula,

$$x = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1+4\epsilon}\right) .$$

[1] You can download a Maple worksheet from the course webpage which will help you to verify your algebra and check your homework answers.

Expanding these two solutions (for ϵ small):

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots\,, \\ -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots\,. \end{cases}$$

We notice that as $\epsilon \to 0$, the second *singular* root 'disappears off to negative infinity'.

Iteration method. In order to retain the second solution to (2.8) as $\epsilon \to 0$ and keep track of its asymptotic behaviour, we must keep the term ϵx^2 as a significant main term in the equation. This means that x must be large. Note that at leading order, the '-1' term in the equation will therefore be negligible compared to the other two terms, i.e. we have

$$\epsilon x^2 + x \approx 0 \quad \Rightarrow \quad x \approx -\frac{1}{\epsilon}\,. \tag{2.9}$$

This suggests the following sensible rearrangement of (2.8),

$$x = -\frac{1}{\epsilon} + \frac{1}{\epsilon x}\,,$$

and hence the iterative scheme

$$x_{n+1} = -\frac{1}{\epsilon} + \frac{1}{\epsilon x_n}\,,$$

with

$$x_0 = -\frac{1}{\epsilon}\,.$$

Note that in this case

$$f(x) = -\frac{1}{\epsilon} + \frac{1}{\epsilon x}\,.$$

Hence

$$\begin{aligned} |f'(x)| &= \left|-\frac{1}{\epsilon x^2}\right| \\ &= \frac{1}{\epsilon}\cdot\frac{1}{x^2} \\ &\approx \epsilon\,, \end{aligned}$$

when $x \approx -1/\epsilon$. Therefore, since ϵ is small, $|f'(x)|$ is small when x is close to the root, and further, we expect an order ϵ improvement in accuracy with each iteration. The first two steps in the iterative process reveals

$$x_1 = -\frac{1}{\epsilon} - 1\,,$$

and

$$\begin{aligned} x_2 &= -\frac{1}{\epsilon} - \frac{1}{1+\epsilon} \\ &= -\frac{1}{\epsilon} - 1 + \epsilon + \cdots\,. \end{aligned}$$

Expansion method. To determine the asymptotic behaviour of the singular root by the expansion method, we simply pose a formal power series expansion for the solution $x(\epsilon)$ starting with an ϵ^{-1} term instead of the usual ϵ^0 term:

$$x(\epsilon) = \frac{1}{\epsilon}x_{-1} + x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots . \tag{2.10}$$

Using this ansatz[2], i.e. substituting (2.10) into (2.8) and equating powers of ϵ^{-1}, ϵ^0 and ϵ^1 etc. generates equations which can be solved for x_{-1}, x_0 and x_1 etc. and thus we can write down the appropriate power series expansion for the singular root.

Rescaling method. There is a more elegant technique for dealing with singular perturbation problems. This involves *rescaling* the variables before posing a formal power series expansion. For example, for the quadratic equation (2.8), set

$$x = \frac{X}{\epsilon} ,$$

and substitute this into (2.8) and multiply through by ϵ,

$$X^2 + X - \epsilon = 0 . \tag{2.11}$$

This is now a *regular perturbation problem.* Hence the problem of finding the appropriate starting point of a trial expansion for a singular perturbation problem is transformed into the problem of finding the appropriate rescaling that regularizes the singular problem. We can now apply the standard methods we have learned thusfar to (2.11), remembering to substitute back $X = \epsilon x$ at the very end to get the final answer.

Note that a practical way to determine the appropriate rescaling to try is to use arguments analogous to those that lead to (2.9) above.

2.2.1. Example (singular perturbation problem). Use an appropriate power series expansion to find an asymptotic approximation as $\epsilon \to 0^+$, correct to $\mathcal{O}(\epsilon^2)$, for the two *small* roots of

$$\epsilon x^3 + x^2 + 2x - 3 = 0 .$$

Then by using a suitable rescaling, find the first three terms of an asymptotic expansion as $\epsilon \to 0^+$ of the singular root.

Example (transcendental equation). Consider the problem of finding the first few terms of a suitable asymptotic approximation to the real large solution of the transcendental equation

$$\epsilon \, x e^x = 1 , \tag{2.12}$$

where $0 < \epsilon \ll 1$. First we should get an idea of how the functions $\epsilon \, x e^x$ and 1 in (2.12) behave. In particular we should graph the function on the left-hand side $\epsilon \, x e^x$ as a function of x and see where its graph crosses the graph of the constant function 1 on the right-hand side. There is clearly only one solution,

[2] Ansatz is German for "approach".

which will be positive, and also large when ϵ is small. In fact when $0 < \epsilon \ll 1$, then

$$xe^x = \frac{1}{\epsilon} \gg 1 \quad \Rightarrow \quad x \gg 1\,,$$

confirming that we expect the root to be large. The question is how large, or more precisely, exactly how does the root scale with ϵ?

Given that the dominant term in (2.12) is e^x, taking the logarithm of both sides of the equation (2.12) might clarify the scaling issue.

$$\begin{aligned}
\Rightarrow \qquad & x + \ln x + \ln \epsilon = 0 \\
\Leftrightarrow \qquad & x = -\ln \epsilon - \ln x \\
\Leftrightarrow \qquad & x = \ln\left(\tfrac{1}{\epsilon}\right) - \ln x\,,
\end{aligned}$$

where in the last step we used that $-\ln A \equiv \ln\left(\frac{1}{A}\right)$. Now we see that when $0 < \epsilon \ll 1$ so that $x \gg 1$, then $x \gg \ln x$ and the root must lie near to $\ln\left(\frac{1}{\epsilon}\right)$, i.e.

$$x \approx \ln\left(\tfrac{1}{\epsilon}\right)\,.$$

This suggests the iterative scheme

$$x_{n+1} = \ln\left(\tfrac{1}{\epsilon}\right) - \ln x_n\,,$$

with $x_0 = \ln\left(\frac{1}{\epsilon}\right)$. Note that in this case we can identify $f(x) \equiv \ln\left(\frac{1}{\epsilon}\right) - \ln x$, and

$$\begin{aligned}
\left|f'(x)\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\tfrac{1}{\epsilon}\right) - \ln x\right)\right| \\
&= \left|-\frac{1}{x}\right| \\
&= \frac{1}{|x|} \\
&\approx \frac{1}{\ln\left(\frac{1}{\epsilon}\right)}\,,
\end{aligned}$$

when x is close to the root. Therefore $|f'(x)|$ is small since ϵ is small. Another good reason for choosing the iteration method here is that a natural expansion sequence is not at all obvious. The iteration scheme gives

$$x_1 = \ln\left(\tfrac{1}{\epsilon}\right) - \ln\ln\left(\tfrac{1}{\epsilon}\right)\,.$$

Then

$$\begin{aligned}
x_2 &= \ln\left(\tfrac{1}{\epsilon}\right) - \ln x_1 \\
&= \ln\left(\tfrac{1}{\epsilon}\right) - \ln\left(\ln\left(\tfrac{1}{\epsilon}\right) - \ln\ln\left(\tfrac{1}{\epsilon}\right)\right) \\
&= \ln\left(\tfrac{1}{\epsilon}\right) - \ln\left(\ln\left(\tfrac{1}{\epsilon}\right)\left(1 - \frac{\ln\ln\left(\frac{1}{\epsilon}\right)}{\ln\left(\frac{1}{\epsilon}\right)}\right)\right) \\
&= \ln\left(\tfrac{1}{\epsilon}\right) - \ln\ln\left(\tfrac{1}{\epsilon}\right) - \ln\left(1 - \frac{\ln\ln\left(\frac{1}{\epsilon}\right)}{\ln\left(\frac{1}{\epsilon}\right)}\right)\,,
\end{aligned}$$

where in the last step we used that $\ln AB \equiv \ln A + \ln B$. Hence, using the Taylor series expansion for $\ln(1-x)$, i.e. $\ln(1-x) \approx -x$, we see that as $\epsilon \to 0^+$,

$$x \sim \ln\left(\tfrac{1}{\epsilon}\right) - \ln\ln\left(\tfrac{1}{\epsilon}\right) + \frac{\ln\ln\left(\tfrac{1}{\epsilon}\right)}{\ln\left(\tfrac{1}{\epsilon}\right)} + \cdots .$$

CHAPTER 3

Asymptotic series

3.1. Asymptotic vs convergent series

Example (the exponential integral). This nicely demonstrates the difference between convergent and asymptotic series. Consider the exponential integral function defined for $x > 0$ by

$$\mathrm{Ei}(x) \equiv \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t\,.$$

Let us look for an analytical approximation to $\mathrm{Ei}(x)$ for $x \gg 1$. Repeatedly integrating by parts gives

$$\begin{aligned}
\mathrm{Ei}(x) &= \left[-\frac{\mathrm{e}^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{\mathrm{e}^{-t}}{t^2}\,\mathrm{d}t \\
&= \frac{\mathrm{e}^{-x}}{x} - \int_x^\infty \frac{\mathrm{e}^{-t}}{t^2}\,\mathrm{d}t \\
&= \frac{\mathrm{e}^{-x}}{x} + \left[\frac{\mathrm{e}^{-t}}{t^2}\right]_x^\infty + 2\int_x^\infty \frac{\mathrm{e}^{-t}}{t^3}\,\mathrm{d}t \\
&= \frac{\mathrm{e}^{-x}}{x} - \frac{\mathrm{e}^{-x}}{x^2} + 2\int_x^\infty \frac{\mathrm{e}^{-t}}{t^3}\,\mathrm{d}t \\
&\;\;\vdots \\
&= \underbrace{\mathrm{e}^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \cdots + (-1)^{N-1}\frac{(N-1)!}{x^N}\right)}_{S_N(x)} + \underbrace{(-1)^N N! \int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t}_{R_N(x)}\,.
\end{aligned}$$

Here we set $S_N(x)$ to be the partial sum of the first N terms,

$$S_N(x) \equiv \mathrm{e}^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \cdots + (-1)^{N-1}\frac{(N-1)!}{x^N}\right)$$

and $R_N(x)$ to be the remainder after N terms

$$R_N(x) \equiv (-1)^N N! \int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t\,.$$

The series for which $S_N(x)$ is the partial sum is *divergent* for any fixed x; notice that for large N the magnitude of the Nth term increases as N increases! Of course $R_N(x)$ is also unbounded as $N \to \infty$, since $S_N(x) + R_N(x)$ must be bounded because $\mathrm{Ei}(x)$ is defined (and bounded) for all $x > 0$.

Suppose we consider N *fixed* and let x become large:

$$
\begin{aligned}
\big|R_N(x)\big| &= \left|(-1)^N N! \int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t\right| \\
&= \big|(-1)^N\big| N! \int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t \\
&= N! \int_x^\infty \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t \\
&< \frac{N!}{x^{N+1}} \int_x^\infty \mathrm{e}^{-t}\,\mathrm{d}t \\
&= \frac{N!}{x^{N+1}}\mathrm{e}^{-x}\,,
\end{aligned}
$$

which tends to zero very rapidly as $x \to \infty$. Note that the ratio of $R_N(x)$ to the last term in $S_N(x)$ is

$$
\begin{aligned}
\left|\frac{R_N(x)}{(N-1)!\mathrm{e}^{-x}x^{-N}}\right| &= \frac{|R_N(x)|}{(N-1)!\mathrm{e}^{-x}x^{-N}} \\
&< \frac{N!\mathrm{e}^{-x}x^{-(N+1)}}{(N-1)!\mathrm{e}^{-x}x^{-N}} \\
&= \frac{N}{x}\,, \qquad (3.1)
\end{aligned}
$$

which also tends to zero as $x \to \infty$. Thus

$$
\mathrm{Ei}(x) = S_N(x) + o\big(\text{last term in } S_N(x)\big)
$$

as $x \to \infty$. In particular, if x is sufficiently large and N fixed, $S_N(x)$ gives a good approximation to $\mathrm{Ei}(x)$; the accuracy of the approximation increases as x increases for N fixed. In fact, as we shall see, this means we can write

$$
\mathrm{Ei}(x) \sim \mathrm{e}^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} + \cdots\right)\,,
$$

as $x \to \infty$.

Note that for x sufficiently large, the terms in $S_N(x)$ will successively decrease initially—for example $2!x^{-3} < x^{-2}$ for x large enough. However at some value $N = N_*(x)$, the terms in $S_N(x)$ for $N > N_*$ will start to increase successively for a given x (however large) because the Nth term,

$$
(-1)^{N-1}\mathrm{e}^{-x}\frac{(N-1)!}{x^N}\,,
$$

is unbounded as $N \to \infty$.

Hence for a given x, there is an optimal value $N = N_*(x)$ for which the greatest accuracy is obtained. Our estimate (3.1) suggests we should take N_* to be the largest integral part of the given x.

In practical terms, such an asymptotic expansion can be of more value than a slowly converging expansion. Asymptotic expansions which give divergent series, can be remarkably accurate: for $\mathrm{Ei}(x)$ with $x = 10$; $N_* = 10$, but $S_4(10)$ approximates $\mathrm{Ei}(10)$ with an error of less than 0.003%.

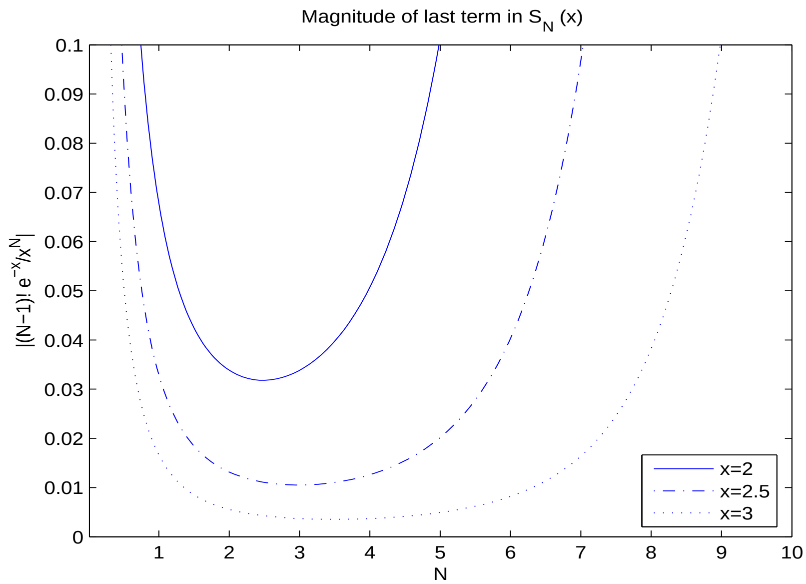


FIGURE 1. The behaviour of the magnitude of the last term in $S_N(x)$, i.e. $\big|(-1)^{N-1}(N-1)!\mathrm{e}^{-x}x^{-N}\big|$, as a function of N for different values of x.

Basic idea. Consider the following power series expansion about $z = z_0$:

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n\,. \tag{3.2}$$

- Such a *power series* is convergent for $|z-z_0| < r$, for some $r \geq 0$, provided (see the Remainder Theorem)

  $$R_N(x) = \sum_{n=N+1}^{\infty} a_n(z-z_0)^n \to 0\,,$$

  as $N \to \infty$ for each *fixed* z satisfying $|z-z_0| < r$.

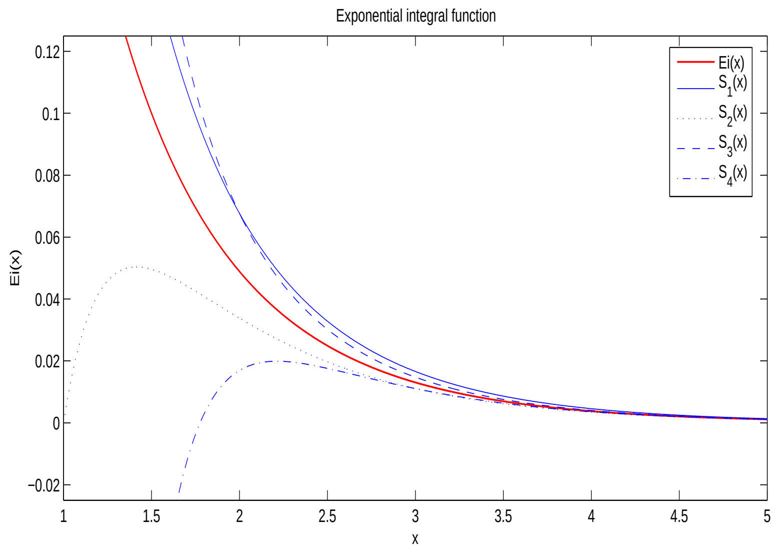
- A function $f(z)$ has an asymptotic series expansion of the form (3.2) as $z \to z_0$, i.e.

  $$f(z) \sim \sum_{n=0}^{\infty} a_n(z-z_0)^n\,, \tag{3.3}$$

  provided

  $$R_N(x) = \mathrm{o}\big((z-z_0)^N\big)\,,$$

  as $z \to z_0$, for each *fixed* N.



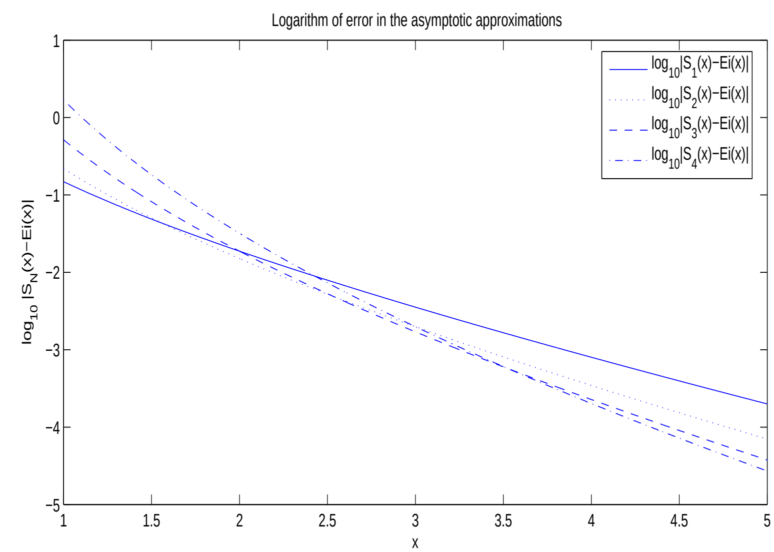


FIGURE 2. In the top figure we show the behaviour of the Exponential Integral function $\mathrm{Ei}(x)$ and four successive asymptotic approximations. The lower figure shows how the magnitude of the difference between the four asymptotic approximations and $\mathrm{Ei}(x)$ varies with x—i.e. the error in the approximations is shown (on a semi-log scale).

3.2. Asymptotic expansions

DEFINITION. *A sequence of gauge functions* $\{\phi_n(x)\}$, $n = 1, 2, \dots$ *is said to form an* asymptotic sequence *as* $x \to x_0$ *if, for all* n,

$$\phi_{n+1}(x) = o\big(\phi_n(x)\big) ,$$

as $x \to x_0$.

Examples. $(x - x_0)^n$ as $x \to x_0$; x^{-n} as $x \to \infty$; $(\sin x)^n$ as $x \to 0$.

DEFINITION. *If* $\{\phi_n(x)\}$ *is an asymptotic sequence of functions as* $x \to x_0$, *we say that*

$$\sum_{n=1}^{\infty} a_n \phi_n(x)$$

where the a_n *are constants, is an* asymptotic expansion *(or asymptotic approximation) of* $f(x)$ *as* $x \to x_0$ *if for each* N

$$f(x) = \sum_{n=1}^{N} a_n \phi_n(x) + o\big(\phi_N(x)\big) , \tag{3.4}$$

as $x \to x_0$, *i.e. the error is asymptotically smaller than the last term in the expansion.*

Remark. An equivalently property to (3.4) is

$$f(x) = \sum_{n=1}^{N-1} a_n \phi_n(x) + \mathcal{O}\big(\phi_N(x)\big) ,$$

as $x \to x_0$ (which can be seen by using that the guage functions $\phi_n(x)$ form an asymptotic sequence).

Notation. We denote an asymptotic expansion by

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x) ,$$

as $x \to x_0$.

DEFINITION. *If the gauge functions form a power sequence, then the asymptotic expansion is called an* asymptotic power series.

Examples. x^n as $x \to 0$; $(x - x_0)^n$ as $x \to x_0$; x^{-n} as $x \to \infty$.

3.3. Properties of asymptotic expansions

Uniqueness. For a *given* asymptotic sequence $\{\phi_n(x)\}$, the asymptotic expansion of $f(x)$ is *unique*; i.e. the a_n are uniquely determined as follows

$$\begin{aligned} a_1 &= \lim_{x\to x_0} \frac{f(x)}{\phi_1(x)}\,; \\ a_2 &= \lim_{x\to x_0} \frac{f(x)-a_1\phi_1(x)}{\phi_2(x)}\,; \\ &\vdots \\ a_N &= \lim_{x\to x_0} \frac{f(x)-\sum_{n=1}^{N-1} a_n\phi_n(x)}{\phi_N(x)}\,; \end{aligned}$$

and so forth.

Non-uniqueness (for a given function). A given function $f(x)$ may have many different asymptotic expansions. For example as $x \to 0$,

$$\begin{aligned} \tan x &\sim x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\ &\sim \sin x + \tfrac{1}{2}(\sin x)^3 + \tfrac{3}{8}(\sin x)^5 + \cdots\,. \end{aligned}$$

Subdominance. An asymptotic expansion may be the asymptotic expansion of more than one function. For example, if as $x \to x_0$

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n\,,$$

then also

$$f(x) + \mathrm{e}^{-\frac{1}{(x-x_0)^2}} \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n\,,$$

as $x \to x_0$ because $\mathrm{e}^{-\frac{1}{(x-x_0)^2}} = \mathrm{o}\big((x-x_0)^n\big)$ as $x \to x_0$ for all n.

In fact,

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

is asymptotic as $x \to x_0$ to any function which differs from $f(x)$ by a function $g(x)$ so long as $g(x) \to 0$ as $x \to x_0$ faster than all powers of $x - x_0$. Such a function $g(x)$ is said to be *subdominant* to the asymptotic power series; the asymptotic power series of $g(x)$ would be

$$g(x) \sim \sum_{n=0}^{\infty} 0 \cdot (x-x_0)^n\,.$$

Hence an asymptotic expansion is asymptotic to a *whole class of functions* that differ from each other by subdominant functions.

Example (subdominance: exponentially small errors). The function e^{-x} is subdominant to an asymptotic power series of the form

$$\sum_{n=0}^{\infty} a_n x^{-n}$$

as $x \to +\infty$; and so if a function $f(x)$ has such a asymptotic expansion, so does $f(x) + \mathrm{e}^{-x}$, i.e. $f(x)$ has such an asymptotic power series expansion up to *exponentially small errors*.

Equating coefficients. If we write

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \sim \sum_{n=0}^{\infty} b_n \cdot (x - x_0)^n \tag{3.5}$$

we mean that the class of functions to which

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n \cdot (x - x_0)^n$$

are asymptotic as $x \to x_0$, are the same. Further, uniqueness of asymptotic expansions means that $a_n = b_n$ for all n, i.e. we may equate coefficients of like powers of $x - x_0$ in (3.5).

Arithmetical operations. Suppose as $x \to x_0$,

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{and} \quad g(x) \sim \sum_{n=0}^{\infty} b_n \phi_n(x)$$

then as $x \to x_0$,

$$\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (a_n + b_n) \cdot \phi_n(x)\,,$$

where α and β are constants. Asymptotic expansions can also be multiplied and divided—perhaps based on an enlarged asymptotic sequence (which we will need to be able to order). In particular for asymptotic power series, when $\phi_n(x) = (x - x_0)^n$, these operations are straightforward:

$$f(x) \cdot g(x) \sim \sum_{n=0}^{\infty} c_n (x - x_0)^n\,,$$

where $c_n = \sum_{m=0}^{n} a_n b_{n-m}$ and if $b_0 \neq 0$, $d_0 = a_0/b_0$, then

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} d_n (x - x_0)^n\,,$$

where $d_n = \left(a_n - \sum_{m=0}^{n-1} d_m b_{n-m}\right)/b_0$.

Integration. An asymptotic power series can be integrated term by term (if $f(x)$ is integrable near $x = x_0$) resulting in the correct asymptotic expansions for the integral. Hence, if

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

as $x \to x_0$, then

$$\int_{x_0}^{x} f(t)\, \mathrm{d}t \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} .$$

Differentiation. Asymptotic expansions cannot in general be differentiated term by term. The problem with differentiation is connected with subdominance. For instance, the two functions

$$f(x) \qquad \text{and} \qquad g(x) = f(x) + \mathrm{e}^{-\frac{1}{(x-x_0)^2}} \sin\left(\mathrm{e}^{\frac{1}{(x-x_0)^2}}\right)$$

differ by a subdominant function and thus have the same asymptotic power series expansion as $x \to x_0$. However $f'(x)$ and

$$g'(x) = f'(x) - 2(x-x_0)^{-3} \cos\left(\mathrm{e}^{\frac{1}{(x-x_0)^2}}\right) + 2(x-x_0)^{-3} \mathrm{e}^{-\frac{1}{(x-x_0)^2}} \sin\left(\mathrm{e}^{\frac{1}{(x-x_0)^2}}\right)$$

do not have the same asymptotic power series expansion as $x \to x_0$.

However if $f'(x)$ exists, is integrable, and as $x \to x_0$,

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n ,$$

then

$$f'(x) \sim \sum_{n=0}^{\infty} n \cdot a_n \cdot (x - x_0)^{n-1} .$$

In particular, if $f(x)$ is *analytic* in some domain, then one can differentiate an asymptotic expansion for $f(x)$ term by term—recall that a real function $f(x)$ is said to be *analytic* at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with a non-zero radius of convergence. For example,

$$\frac{1}{x-1} \sim \frac{1}{x} + \frac{1}{x^2} + \cdots ,$$

as $x \to +\infty$ implies, since the power series shown is in fact convergent for every $x > 1$ and therefore $\frac{1}{x-1}$ is analytic for all $x > 1$, that

$$\frac{1}{(x-1)^2} \sim \frac{1}{x^2} + \frac{2}{x^3} + \cdots ,$$

as $x \to +\infty$. (Both of the power series are the Taylor series expansions for the respective functions shown.)

3.4. Asymptotic expansions of integrals

Integral representations. When modelling many physical phenomena, it is often useful to know the asymptotic behaviour of integrals of the form

$$I(x) = \int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t\,, \tag{3.6}$$

as $x \to x_0$ (or more generally, the integral of a complex function along a contour). For example, many functions have integral representations:

- the *error function*,
$$\mathrm{Erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t;$$
- the *incomplete gamma function* ($x > 0$, $a > 0$),
$$\gamma(a,x) \equiv \int_0^x \mathrm{e}^{-t}\,t^{a-1}\,\mathrm{d}t\,.$$

Many other special functions such as the *Bessel*, *Airy* and *hypergeometric functions* have integral representations as they are solutions of various classes of differential equations. Also, if we use *Laplace*, *Fourier* or *Hankel transformations* to solve differential equations, we are often left with an integral representation of the solution (eg. to determine an inverse Laplace transform we must evaluate a *Bromwich contour integral*). Two simple techniques for obtaining asymptotic expansions of integrals like (3.6) are demonstrated through the following two examples.

Example (Inserting an asymptotic expansion of the integrand). We can obtain an asymptotic expansion of the incomplete gamma function $\gamma(a,x)$ as $x \to 0$, by expanding the integrand in powers of t and integrating term by term.

$$\begin{aligned}
\gamma(a,x) &= \int_0^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) t^{a-1}\,\mathrm{d}t \\
&= \frac{x^a}{a} - \frac{x^{a+1}}{(a+1)} + \frac{x^{a+2}}{(a+2)2!} - \frac{x^{a+3}}{(a+3)3!} + \cdots \\
&= x^a \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(a+n)n!}\,,
\end{aligned}$$

which in fact converges for all x. However for large values of x, it converges very slowly which in practical terms is not very useful.

Example (Integrating by parts). For large x we can obtain an asymptotic expansion of $\gamma(a,x)$ as $x \to +\infty$ as follows. Write

$$\gamma(a,x) = \underbrace{\int_0^\infty \mathrm{e}^{-t}\,t^{a-1}\,\mathrm{d}t}_{\Gamma(a)} - \underbrace{\int_x^\infty \mathrm{e}^{-t}\,t^{a-1}\,\mathrm{d}t}_{\mathrm{Ei}(1-a,x)}\,.$$

Recall that the (complete) gamma function $\Gamma(a)$ is defined by

$$\Gamma(a) \equiv \int_0^\infty \mathrm{e}^{-t}\, t^{a-1}\, \mathrm{d}t\,.$$

And a generalization of the exponential integral above is labelled ($x > 0$)

$$\mathrm{Ei}(1-a,x) \equiv \int_x^\infty \mathrm{e}^{-t}\, t^{a-1}\, \mathrm{d}t\,.$$

We now integrate $\mathrm{Ei}(1-a,x)$ by parts successively:

$$\begin{aligned}
\mathrm{Ei}(1-a,x) &= \mathrm{e}^{-x} x^{a-1} + (a-1)\int_x^\infty \mathrm{e}^{-t}\, t^{a-2}\, \mathrm{d}t \\
&\;\vdots \\
&= \mathrm{e}^{-x}\left(x^{a-1} + (a-1)x^{a-2} + \cdots + (a-1)\cdots(a-N+1)x^{a-N}\right) \\
&\quad + (a-1)(a-2)\cdots(a-N)\int_x^\infty \mathrm{e}^{-t}\, t^{a-N-1}\, \mathrm{d}t.
\end{aligned}$$

Note that for any fixed $N > a-1$,

$$\begin{aligned}
\left|\int_x^\infty \mathrm{e}^{-t}\, t^{a-N-1}\, \mathrm{d}t\right| &\le \int_x^\infty \left|\mathrm{e}^{-t}\, t^{a-N-1}\right|\, \mathrm{d}t \\
&= \int_x^\infty \mathrm{e}^{-t}\, t^{a-N-1}\, \mathrm{d}t \\
&\le x^{a-N-1}\int_x^\infty \mathrm{e}^{-t}\, \mathrm{d}t \\
&= x^{a-N-1}\mathrm{e}^{-x} \\
&= \mathrm{o}(x^{a-N}\mathrm{e}^{-x})
\end{aligned}$$

as $x \to +\infty$, and so the expansion for $\mathrm{Ei}(1-a,x)$ above is asymptotic. Hence

$$\gamma(a,x) \sim \Gamma(a) - \mathrm{e}^{-x}\, x^a\left(\frac{1}{x} + \frac{a-1}{x^2} + \frac{(a-1)(a-2)}{x^3} + \cdots\right),$$

as $x \to +\infty$, although the series is divergent. Note further that if $R_N(x)$ is the remainder after N terms (and with $N > a-1$):

$$\begin{aligned}
\frac{\left|R_N(x)\right|}{\left|\mathrm{e}^{-x}\, x^{a-N}(a-1)\cdots(a-N+1)\right|} &\le \frac{\mathrm{e}^{-x}\, x^{a-N-1}\left|(a-1)\cdots(a-N)\right|}{\mathrm{e}^{-x}\, x^{a-N}\left|(a-1)\cdots(a-N+1)\right|} \\
&= \frac{\left|a-N\right|}{x}\,.
\end{aligned}$$

Hence for a given x, only use the largest number of N (integer) terms in the asymptotic expansion such that $x > |a-N|$.

CHAPTER 4

Laplace integrals

A *Laplace integral* has the form

$$I(x) = \int_a^b f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t \tag{4.1}$$

where we assume $x > 0$. Typically x is a large parameter and we are interested in the asymptotic behaviour of $I(x)$ as $x \to +\infty$. Note that we can write $I(x)$ in the form

$$I(x) = \frac{1}{x} \int_a^b \frac{f(t)}{\phi'(t)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{x\phi(t)} \right)\, \mathrm{d}t\,.$$

Integrating by parts gives

$$I(x) = \underbrace{\left[\frac{1}{x} \cdot \frac{f(t)}{\phi'(t)} \cdot \mathrm{e}^{x\phi(t)} \right]_a^b}_{\text{boundary term}} - \underbrace{\frac{1}{x} \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{f(t)}{\phi'(t)} \right) \cdot \mathrm{e}^{x\phi(t)}\, \mathrm{d}t}_{\text{integral term}}\,. \tag{4.2}$$

If the integral term is asymptotically smaller than the boundary term, i.e.

$$\text{integral term} = \mathrm{o}(\text{boundary term})$$

as $x \to +\infty$, then

$$I(x) \sim \left[\frac{1}{x} \cdot \frac{f(t)}{\phi'(t)} \cdot \mathrm{e}^{x\phi(t)} \right]_a^b$$

as $x \to +\infty$, i.e.

$$I(x) \sim \frac{1}{x} \cdot \frac{f(b)}{\phi'(b)} \cdot \mathrm{e}^{x\phi(b)} - \frac{1}{x} \cdot \frac{f(a)}{\phi'(a)} \cdot \mathrm{e}^{x\phi(a)} \tag{4.3}$$

and we have a useful asymptotic approximation for $I(x)$ as $x \to +\infty$. In general, this will in fact be the case, i.e. (4.3) is valid, if $\phi(t)$, $\phi'(t)$ and $f(t)$ are continuous functions (possibly complex) and the following condition is satisfied:

$$\phi'(t) \neq 0 \text{ for } a \leq t \leq b \text{ and either } f(a) \neq 0 \text{ or } f(b) \neq 0\,.$$

This ensures that the integral term in (4.2) is bounded and is asymptotically smaller than the boundary term. Further, we may also continue to integrate by parts to generate further terms in the asymptotic expansion of $I(x)$; each integration by parts generates a new factor of $1/x$.

4.1. Laplace's method

We can see that for Laplace integrals, integration by parts may fail for example, when $\phi'(t)$ has a zero somewhere in $a \leq t \leq b$. *Laplace's method* is a general technique that allows us to generate an asymptotic expansion for Laplace integrals for large x (and in particular when integration by parts fails). Recall

$$I(x) = \int_a^b f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t$$

where we now suppose $f(t)$ and $\phi(t)$ are real, continuous functions.

Basic idea. If $\phi(t)$ has a global maximum at $t = c$ with $a \leq c \leq b$ and if $f(c) \neq 0$, then it is only the neighbourhood of $t = c$ that contributes to the full asymptotic expansion of $I(x)$ as $x \to +\infty$.

Procedure.
Step 1. We may approximate $I(x)$ by $I(x;\epsilon)$ where

$$I(x;\epsilon) = \begin{cases} \displaystyle\int_{c-\epsilon}^{c+\epsilon} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t, & \text{if } a < c < b, \\[2ex] \displaystyle\int_{a}^{a+\epsilon} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t, & \text{if } c = a, \\[2ex] \displaystyle\int_{b-\epsilon}^{b} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t, & \text{if } c = b, \end{cases} \tag{4.4}$$

where $\epsilon > 0$ is *arbitrary*, but sufficiently small to guarantee that each of the subranges of integration indicated are contained in the interval $[a, b]$. Such a step is valid if the asymptotic expansion of $I(x;\epsilon)$ as $x \to +\infty$ does not depend on ϵ *and* is identical to the asymptotic expansion of $I(x)$ as $x \to +\infty$. Both of these results are in fact true since (eg. when $a < c < b$) the terms

$$\left| \int_a^{c-\epsilon} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t \right| + \left| \int_{c+\epsilon}^{b} f(t)\, \mathrm{e}^{x\phi(t)}\, \mathrm{d}t \right|$$

are subdominant to $I(x)$ as $x \to +\infty$. This is because $\mathrm{e}^{x\phi(t)}$ is exponentially small compared to $\mathrm{e}^{x\phi(c)}$ for $a \leq t \leq c - \epsilon$ and $c + \epsilon \leq t \leq b$. In other words, changing the limits of integration only introduces *exponentially small errors* (all this can be rigorously proved by integrating by parts). Hence we simply replace $I(x)$ by the truncated integral $I(x;\epsilon)$.

Step 2. Now $\epsilon > 0$ can be chosen small enough so that (now we're confined to the narrow region surrounding $t = c$) it is valid to replace $\phi(t)$ by the first few terms in its Taylor or asymptotic series expansion.

- If $\phi'(c) = 0$ with $a \leq c \leq b$ and $\phi''(c) \neq 0$, approximate $\phi(t)$ by

$$\phi(t) \approx \phi(c) + \frac{1}{2}\phi''(c) \cdot (t-c)^2\,,$$

and approximate $f(t)$ by

$$f(t) \approx f(c) \neq 0. \tag{4.5}$$

- If $c = a$ or $c = b$ and $\phi'(c) \neq 0$, approximate $\phi(t)$ by

  $$\phi(t) \approx \phi(c) + \phi'(c) \cdot (t - c)\,,$$

  and approximate $f(t)$ by

  $$f(t) \approx f(c) \neq 0. \tag{4.6}$$

Step 3. Having substituted the approximations for ϕ and f indicated above, we now extend the endpoints of integration to infinity, in order to evaluate the resulting integrals (again this only introduces exponentially small errors).

- If $\phi'(c) = 0$ with $a < c < b$, we must have $\phi''(c) < 0$ ($t = c$ is a maximum) and so as $x \to +\infty$,

  $$\begin{aligned} I(x) &\sim \int_{c-\epsilon}^{c+\epsilon} f(c)\mathrm{e}^{x\left(\phi(c)+\frac{1}{2}\phi''(c)\cdot(t-c)^2\right)}\,\mathrm{d}t \\ &\sim f(c)\mathrm{e}^{x\phi(c)} \int_{-\infty}^{\infty} \mathrm{e}^{x\cdot\frac{\phi''(c)}{2}(t-c)^2}\,\mathrm{d}t \\ &= \frac{\sqrt{2}f(c)\mathrm{e}^{x\phi(c)}}{\sqrt{-x\phi''(c)}} \int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s \end{aligned} \tag{4.7}$$

  where in the last step we made the substitution

  $$s = +\sqrt{-x \cdot \frac{\phi''(c)}{2}}(t - c)\,.$$

  Since (see the formula sheet in Appendix B)

  $$\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}\,,$$

  we get

  $$I(x) \sim \frac{\sqrt{2\pi}f(c)\mathrm{e}^{x\phi(c)}}{\sqrt{-x\phi''(c)}}\,, \tag{4.8}$$

  as $x \to +\infty$.

  If $\phi'(c) = 0$ and $c = a$ or $c = b$, then the leading order behaviour for $I(x)$ is the same as that in (4.8) except multiplied by a factor $\frac{1}{2}$—when we extend the limits of integration, we only do so in one direction, so that the integral in (4.7) only extends over a semi-infinite range.

- If $c = a$ and $\phi'(c) \neq 0$, we must have $\phi'(c) < 0$, and as $x \to +\infty$,

$$\begin{aligned} I(x) &\sim \int_a^{a+\epsilon} f(a)\mathrm{e}^{x(\phi(a)+\phi'(a)\cdot(t-a))}\,\mathrm{d}t \\ &\sim f(a)\mathrm{e}^{x\phi(a)} \int_0^{\infty} \mathrm{e}^{x\cdot\phi'(a)\cdot(t-a)}\,\mathrm{d}t \\ \Rightarrow \quad I(x) &\sim -\frac{f(a)\mathrm{e}^{x\phi(a)}}{x\phi'(a)}\,. \end{aligned}$$

  If $c = b$ and $\phi'(c) \neq 0$, we must have $\phi'(c) > 0$, and a similar argument implies that as $x \to +\infty$,

$$I(x) \sim \frac{f(b)\mathrm{e}^{x\phi(b)}}{x\phi'(b)}\,.$$

Remarks.

(1) If $\phi(t)$ achieves its global maximum at several points in $[a, b]$, decompose the integral $I(x)$ into several intervals, each containing a single maximum. Perform the analysis above and compare the contributions to the asymptotic behaviour of $I(x)$ (which will be additive) from each subinterval. The final ordering of the asymptotic expansion will then depend on the behaviour of $f(t)$ at the maximal values of $\phi(t)$.

(2) If the maximum is such that $\phi'(c) = \phi''(c) = \cdots = \phi^{(m-1)}(c) = 0$ and $\phi^{(m)}(c) \neq 0$ then use that $\phi(t) \approx \phi(c) + \frac{1}{m!}\phi^{(m)}(c)\cdot(t-c)^m$.

(3) In (4.5) and (4.6) above we assumed $f(c) \neq 0$—see the beginning of this section—the case when $f(c) = 0$ is more delicate and treated in many books—see Bender & Orszag [**2**].

(4) We have only derived the leading order behaviour. It is also possible to determine higher order terms, though the process is much more involved—again see Bender & Orszag [**2**].

Example (Stirling's formula). We shall try to find the leading order behaviour of the (complete) Gamma function

$$\Gamma(x+1) \equiv \int_0^{\infty} \mathrm{e}^{-t}\, t^x\,\mathrm{d}t\,,$$

as $x \to +\infty$. First note that we can write

$$\Gamma(x+1) = \int_0^{\infty} \mathrm{e}^{-t+x\ln t}\,\mathrm{d}t\,.$$

Second we try to convert it more readily to the standard Laplace integral form by making the substitution $t = xr$ (this really has the effect of creating

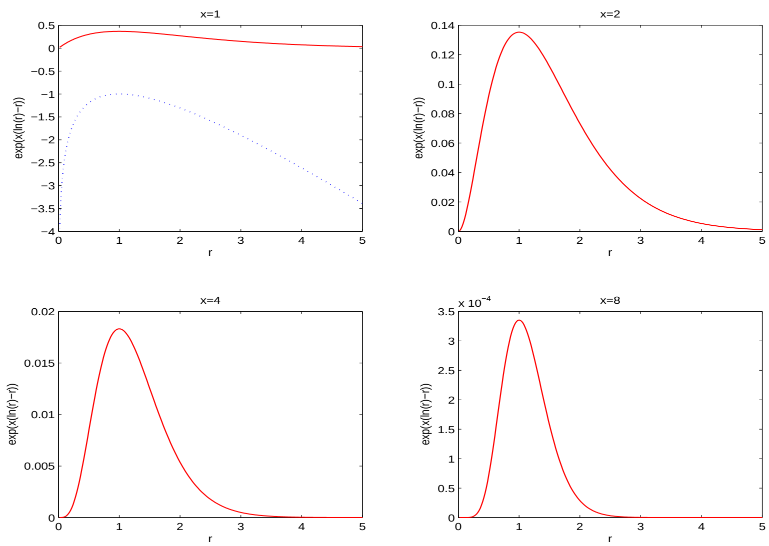


FIGURE 1. Shown are the graphs of $\exp(x(\ln r - r))$ for successively increasing values of x (solid curves); the dotted curve is $\phi(r) = \ln r - r$. We see that it is the region around the global maximum of $\phi(r)$ that contributes most to the integral $\int_0^\infty \exp(x(\ln r - r))\,\mathrm{d}r$.

a global maximum for ϕ),

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty \mathrm{e}^{-xr + x\ln x + x\ln r}\, x\,\mathrm{d}r \\ &= x^{x+1}\int_0^\infty \mathrm{e}^{x(-r+\ln r)}\,\mathrm{d}r\,.\end{aligned}$$

Hence $f(r) \equiv 1$ and $\phi(r) = -r + \ln r$. Since $\phi'(r) = -1 + \frac{1}{r}$ and $\phi''(r) = -\frac{1}{r^2}$, for all $r > 0$, we conclude that ϕ has a local (& global) maximum at $r = 1$. Hence after collapsing the range of integration to a narrow region surrounding $r = 1$, we approximate

$$\begin{aligned}\phi(r) &\approx \phi(1) + \frac{\phi''(1)}{2}\cdot(r-1)^2 \\ &= -1 - \frac{1}{2}\cdot(r-1)^2\end{aligned}$$

Subsequently extending the range of integration out to infinity again we see that

$$\Gamma(x+1) \sim x^{x+1}\int_{-\infty}^{\infty} \mathrm{e}^{-x}\cdot\mathrm{e}^{-\frac{x}{2}(r-1)^2}\,\mathrm{d}r.$$

Making the substitution $s^2 = \frac{x}{2}\cdot(r-1)^2$ and using that $\int_{-\infty}^{\infty}\mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}$, then reveals that as $x \to +\infty$,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\cdot x^x\cdot\mathrm{e}^{-x}\,.$$

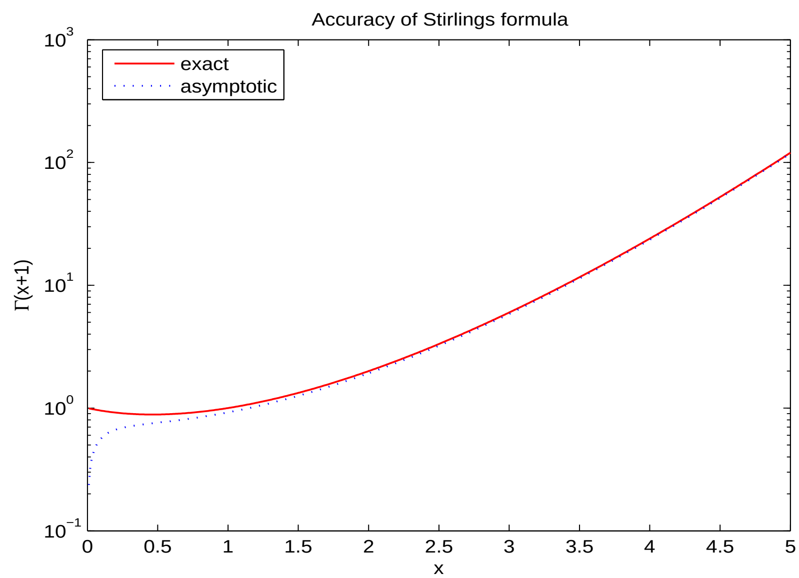


FIGURE 2. On a semi–log plot, we compare the exact values of $\Gamma(x+1)$ with Stirling's formula.

When $x \in \mathbb{N}$, this is *Stirling's formula* for the asymptotic behaviour of the factorial function for large integers.

4.2. Watson's lemma

Based on the ideas above, we can prove a more sophisticated result for a simpler Laplace integral.

WATSON'S LEMMA. *Consider the following example of a Laplace integral (for some $b > 0$)*

$$I(x) \equiv \int_0^b f(t)\, \mathrm{e}^{-xt}\, \mathrm{d}t\,. \tag{4.9}$$

Suppose $f(t)$ is continuous on $[0, b]$ and has the asymptotic expansion as $t \to 0+$,

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n}\,. \tag{4.10}$$

We assume that $\alpha > -1$ and $\beta > 0$ so that the integral is bounded near $t = 0$; if $b = \infty$, we also require that $f(t) = \mathrm{o}(\mathrm{e}^{ct})$ as $t \to +\infty$ for some $c > 0$, to guarantee the integral is bounded for large t. Then as $x \to +\infty$,

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}\,. \tag{4.11}$$

PROOF. The basic idea is to use Laplace's method and to take advantage of the fact that here we have the simple exponent $\phi(t) = -t$ in (4.9) (which has a global maximum at the left end of the range of integration at $t = 0$). This means that we can actually generate all the terms in the asymptotic power series expansion of $I(x)$ in (4.9) by plugging the asymptotic power series for $f(t)$ in (4.10) into (4.9) and proceeding as before.

Step 1. Replace $I(x)$ by $I(x;\epsilon)$ where

$$I(x;\epsilon) = \int_0^\epsilon f(t)\mathrm{e}^{-xt}\,\mathrm{d}t. \tag{4.12}$$

This approximation only introduces exponentially small errors for any $\epsilon > 0$.

Step 2. We can now choose $\epsilon > 0$ small enough so that the first N terms in the asymptotic series for $f(t)$ are a good approximation to $f(t)$, i.e.

$$\left| f(t) - t^\alpha \sum_{n=0}^{N} a_n t^{\beta n} \right| \leq K \cdot t^{\alpha+\beta(N+1)}, \tag{4.13}$$

for $0 \leq t \leq \epsilon$ and some constant $K > 0$. Substituting the first N terms in the series for $f(t)$ into (4.12) we see that, using (4.13),

$$\begin{aligned}
\left| I(x;\epsilon) - \sum_{n=0}^{N} a_n \int_0^\epsilon t^{\alpha+\beta n}\mathrm{e}^{-xt}\,\mathrm{d}t \right| &= \left| \int_0^\epsilon \left(f(t) - t^\alpha \sum_{n=0}^{N} a_n t^{\beta n} \right) \mathrm{e}^{-xt}\,\mathrm{d}t \right| \\
&\leq \int_0^\epsilon \left| f(t) - t^\alpha \sum_{n=0}^{N} a_n t^{\beta n} \right| \mathrm{e}^{-xt}\,\mathrm{d}t \\
&\leq K \cdot \int_0^\epsilon t^{\alpha+\beta(N+1)}\mathrm{e}^{-xt}\,\mathrm{d}t \\
&\leq K \cdot \int_0^\infty t^{\alpha+\beta(N+1)}\mathrm{e}^{-xt}\,\mathrm{d}t\,.
\end{aligned}$$

Hence using the identity

$$\int_0^\infty t^m \mathrm{e}^{-xt}\,\mathrm{d}t \equiv \frac{\Gamma(m+1)}{x^{m+1}} \tag{4.14}$$

we have just established that

$$\left| I(x;\epsilon) - \sum_{n=0}^{N} a_n \int_0^\epsilon t^{\alpha+\beta n}\mathrm{e}^{-xt}\,\mathrm{d}t \right| \leq K \cdot \frac{\Gamma(\alpha+\beta+\beta N+1)}{x^{\alpha+\beta+\beta N+1}}\,.$$

Step 3. Extending the range of integration to $[0,\infty)$ and using the identity (4.14) again, we get that

$$\begin{aligned}
I(x) &= \sum_{n=0}^{N} a_n \int_0^\infty t^{\alpha+\beta n}\mathrm{e}^{-xt}\,\mathrm{d}t + \mathrm{o}\left(\frac{1}{x^{\alpha+\beta N+1}}\right) \\
&= \sum_{n=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} + \mathrm{o}\left(\frac{1}{x^{\alpha+\beta N+1}}\right),
\end{aligned}$$

as $x \to +\infty$. Since this is true for every N, we have proved (4.11) and thus the Lemma. □

Remark. Essentially Watson's lemma tells us that under the conditions outlined: if we substitute the asymptotic expansion (4.10) for $f(t)$ into the integral $I(x)$ and extend the range of integration to $0 \leq t \leq \infty$, then integrating term by term generates the correct asymptotic expansion for $I(x)$ as $x \to +\infty$.

Example. To apply Watson's lemma to the modified Bessel function

$$K_0(x) \equiv \int_1^\infty (s^2-1)^{-\frac{1}{2}}\, \mathrm{e}^{-xs}\, \mathrm{d}s\,,$$

we first substitute $s = t+1$, so the lower endpoint of integration is $t = 0$:

$$K_0(x) = \mathrm{e}^{-x} \int_0^\infty (t^2+2t)^{-\frac{1}{2}}\, \mathrm{e}^{-xt}\, \mathrm{d}t\,.$$

For $|t| < 2$, the binomial theorem implies

$$\begin{aligned}(t^2+2t)^{-\frac{1}{2}} &= (2t)^{-\frac{1}{2}} \left(1+\tfrac{t}{2}\right)^{-\frac{1}{2}} \\ &= (2t)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \left(-\tfrac{t}{2}\right)^n \cdot \frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}\,.\end{aligned}$$

Watson's lemma then immediately tells us that as $x \to +\infty$

$$\begin{aligned}K_0(x) &\sim \mathrm{e}^{-x} \int_0^\infty \sum_{n=0}^{\infty} (2t)^{-\frac{1}{2}} \left(-\tfrac{t}{2}\right)^n \cdot \frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})} \mathrm{e}^{-xt} \mathrm{d}t \\ &= \mathrm{e}^{-x} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})} \frac{(-1)^n}{2^{n+\frac{1}{2}}} \int_0^\infty t^{n-\frac{1}{2}} \mathrm{e}^{-xt} \mathrm{d}t \\ &= \mathrm{e}^{-x} \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(\Gamma(n+\frac{1}{2})\right)^2}{2^{n+\frac{1}{2}} n!\,\Gamma(\frac{1}{2}) x^{n+\frac{1}{2}}}\,.\end{aligned}$$

Note. We can use Watson's lemma to determine the leading order behaviour (and higher orders) of more general Laplace integrals such as (4.1), as $x \to +\infty$. Simply make the change of variables $s = -\phi(t)$ in (4.1) so that

$$I(x) = -\int_{-\phi(a)}^{-\phi(b)} F(s) \mathrm{e}^{-xs}\,, \mathrm{d}s$$

where

$$F(s) = -\frac{f\big(\phi^{-1}(-s)\big)}{\phi'\big(\phi^{-1}(-s)\big)}\,.$$

However, if $t = \phi^{-1}(-s)$ is intricately *multi-valued*, then use the more direct version of Laplace's method—for example if $\phi(t)$ has a global maximum at $t = c$ (for more details see Bender & Orszag [**2**]).

CHAPTER 5

Method of stationary phase

This method was originally developed by Stokes and Kelvin in the 19th century in their study of water waves. Consider the general Laplace integral in which $\phi(t)$ is pure imaginary, i.e. $\phi(t) = \mathrm{i}\psi(t)$ so that

$$I(x) = \int_a^b f(t)\,\mathrm{e}^{\mathrm{i}x\psi(t)}\,\mathrm{d}t,$$

where a, b, x, $f(t)$ and $\psi(t)$ are all real. In this case, $I(x)$ is called the *generalized Fourier integral*. Note that the term $\mathrm{e}^{\mathrm{i}x\psi(t)}$ is purely oscillatory and so we cannot exploit the exponential decay of the integrand away from a maximum as was done in Laplace's method and Watson's lemma. However, if x is large, the integrand in $I(x)$ oscillates rapidly and so we might expect approximate cancellation of positive and negative contributions from adjacent intervals, leading to a small net contribution to the integral. In fact we have the following result.

Riemann-Lebesgue lemma. *If $|f(t)|$ is integrable and $\psi(t)$ is continuously differentiable over $a \leq t \leq b$, but $\psi(t)$ is not constant over any subinterval of $a \leq t \leq b$, then as $x \to +\infty$,*

$$I(x) = \int_a^b f(t)\,\mathrm{e}^{\mathrm{i}x\psi(t)}\,\mathrm{d}t \to 0\,.$$

To obtain an asymptotic expansion of $I(x)$ as $x \to +\infty$, we integrate by parts as before. This is valid provided the boundary terms are finite and the resulting integral exists; for example, provided that $f(t)/\psi'(t)$ is smooth (in particular bounded) over $a \leq t \leq b$ and non-zero at either boundary, then

$$I(x) = \underbrace{\left[\frac{f(t)}{\mathrm{i}x\psi'(t)}\cdot \mathrm{e}^{\mathrm{i}x\psi(t)}\right]_a^b}_{\text{boundary term}} - \underbrace{\frac{1}{\mathrm{i}x}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{\psi'(t)}\right)\mathrm{e}^{\mathrm{i}x\psi(t)}\,\mathrm{d}t}_{\text{integral term}}\,.$$

The integral term is $\mathrm{o}\left(\frac{1}{x}\right)$ as $x \to +\infty$ by the Riemann-Lebesgue lemma and so as $x \to +\infty$,

$$I(x) \sim \left[\frac{f(t)}{\mathrm{i}x\psi'(t)}\cdot \mathrm{e}^{\mathrm{i}x\psi(t)}\right]_a^b\,.$$

Integration by parts may fail if $\psi(t)$ has a *stationary point* in the range of integration, i.e. $\psi'(c) = 0$ for some $a \leq c \leq b$.

Basic idea. If $\psi'(c) = 0$ for some $a \leq c \leq b$ and $\psi'(t) \neq 0$ for all $t \neq c$ in $[a, b]$, then it is the neighbourhood of $t = c$ that generates the leading order asymptotic term in the full asymptotic expansion of $I(x)$ as $x \to +\infty$.

Procedure.

Step 1. For a small $\epsilon > 0$ we decompose $I(x)$ to $I(x;\epsilon)$ just like in (4.4) for Laplace's method. The remainder terms we neglect, for example in the case when $a < c < b$, are

$$\int_a^{c-\epsilon} f(t)\,\mathrm{e}^{\mathrm{i}x\psi(t)}\,\mathrm{d}t + \int_{c+\epsilon}^b f(t)\,\mathrm{e}^{\mathrm{i}x\psi(t)}\,\mathrm{d}t$$

and these vanish like $1/x$ as $x \to +\infty$ because $\psi(t)$ has no stationary points in either interval and we can integrate by parts and apply the Riemann-Lebesgue lemma.

Step 2. With ϵ small enough, to obtain the leading order behaviour we replace $\psi(t)$ and $f(t)$ by the approximations

$$\psi(t) \approx \psi(c) + \frac{\psi''(c)}{2!} \cdot (t-c)^2$$

and

$$f(t) \approx f(c)\,.$$

(This assumes $\psi''(c) \neq 0$, otherwise we must consider higher order terms—see the remarks below). Hence we get as $x \to +\infty$,

$$I(x) \sim \int_{c-\epsilon}^{c+\epsilon} f(c)\,\mathrm{e}^{\mathrm{i}x\cdot\left(\psi(c)+\frac{\psi''(c)}{2!}\cdot(t-c)^2\right)}\,\mathrm{d}t\,.$$

Step 3. We extend the range of integration to infinity in each direction—this again introduces terms which vanish like $1/x$ as $x \to +\infty$ and which will be asymptotically smaller terms that can be neglected. Then using the substitution

$$s = +\sqrt{x \cdot \frac{|\psi''(c)|}{2}} \cdot (t-c)\,,$$

(a change of variables due to Morse) yields

$$I(x) \sim f(c)\,\mathrm{e}^{\mathrm{i}x\psi(c)} \cdot \sqrt{\frac{2}{x|\psi''(c)|}} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\cdot\mathrm{sgn}(\psi''(c))\cdot s^2}\,\mathrm{d}s\,, \tag{5.1}$$

where

$$\mathrm{sgn}(y) \equiv \begin{cases} +1\,, & \text{if } y > 0\,, \\ -1\,, & \text{if } y < 0\,. \end{cases}$$

To evaluate the integral on the right, use that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\pm\mathrm{i}s^2}\,\mathrm{d}s = \sqrt{\pi}\mathrm{e}^{\pm\mathrm{i}\pi/4}$$

so that eventually we get

$$I(x) \sim f(c) \cdot \mathrm{e}^{\mathrm{i}x\psi(c)+\mathrm{i}\cdot\mathrm{sgn}(\psi''(c))\pi/4} \cdot \sqrt{\frac{2\pi}{x|\psi''(c)|}}\,,$$

as $x \to +\infty$.

Remarks.

(1) If $c = a$ or $c = b$, the contribution from the integral, which is only over a semi-infinite interval, means that the asymptotic result above must be multiplied by a factor of $\frac{1}{2}$.

(2) If $\psi(t)$ has many stationary points in $[a, b]$, then we split up the integral into intervals containing only one stationary point and deal with each one independently (though their contributions are additive). Again, the relative size of $f(t)$ at the stationary points of $\psi(t)$ will now be important.

(3) If the stationary point is such that $\psi'(c) = \psi''(c) = \cdots = \psi^{m-1}(c) = 0$ and $\psi^m(c) \neq 0$ then use that $\psi(t) \approx \psi(c) + \frac{\psi^m(c)}{m!} \cdot (t-c)^m$ instead. In this case $I(x)$ behaves like $x^{-1/m}$ as $x \to +\infty$.

(4) Again, a very useful reference for more details is Bender & Orszag [**2**].

Example. To find the leading order asymptotic behaviour as $x \to +\infty$ of

$$I(x) = \int_0^3 t \cos x\big(\tfrac{1}{3}t^3 - t\big)\,\mathrm{d}t\,,$$

we first write the integral in the form

$$I(x) = \mathrm{Re}\left\{\int_0^3 t\,\mathrm{e}^{\mathrm{i}x\left(\frac{1}{3}t^3 - t\right)}\,\mathrm{d}t\right\}$$

so that we can identify $f(t) \equiv t$ and $\psi(t) = \frac{1}{3}t^3 - t$. Hence $\psi'(t) = t^2 - 1$ and $\psi''(t) = 2t$ so that ψ has two stationary points, of which only the positive one lies in the range of integration at $c = 1$. Since $\psi''(t) = 2t > 0$ for $t \geq 0$, this is a local and global minimum. Hence truncating our interval of integration to a small neighbourhood of c (this only introduces asymptotically smaller terms), we then make the approximations

$$\psi(t) \approx \psi(c) + \frac{\psi''(c)}{2} \cdot (t-c)^2\,,$$

and

$$f(t) \approx f(c)\,.$$

Now we extend the endpoints of integration to infinity (again only introducing asymptotically smaller terms) so that as $x \to +\infty$,

$$I(x) \sim f(c)\,\mathrm{e}^{\mathrm{i}x\psi(c)} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x \cdot \frac{\psi''(c)}{2} \cdot (t-c)^2}\,\mathrm{d}t\,.$$

Making the substitution

$$s = +\sqrt{x \cdot \frac{\psi''(c)}{2}} \cdot (t-c)\,,$$

generates

$$I(x) \sim f(c)\,\mathrm{e}^{\mathrm{i}x\psi(c)} \cdot \sqrt{\frac{2}{x\psi''(c)}} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s^2}\,\mathrm{d}s\,.$$

Since (see the formula sheet in Appendix B)

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s^2}\,\mathrm{d}s = \sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\pi/4}\,,$$

we get

$$I(x) \sim f(c)\,\mathrm{e}^{\mathrm{i}(x\psi(c)+\pi/4)} \cdot \sqrt{\frac{2\pi}{x\psi''(c)}}\,.$$

Now taking the real part of this and using that $\psi(c) = -2/3$, $\psi''(c) = 2$ and $f(c) = 1$ we get that as $x \to +\infty$,

$$I(x) \sim \sqrt{\frac{\pi}{x}}\cos\left(\frac{\pi}{4} - \frac{2x}{3}\right)\,.$$

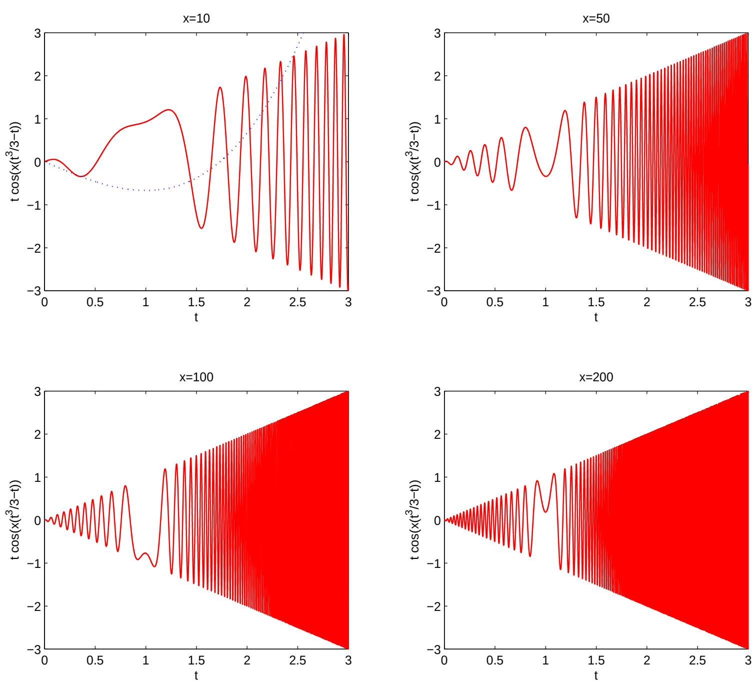


FIGURE 1. This demonstrates why it is the region around the stationary point of the cubic $\frac{1}{3}t^3 - t$ that will contribute most to the integral $\int_0^3 t\cos\left(x\left(\frac{1}{3}t^3 - t\right)\right)\mathrm{d}t$. Shown above are the graphs of $\cos\left(x\left(\frac{1}{3}t^3 - t\right)\right)$ for successively increasing values of x (solid curves); the dotted curve is the cubic $\frac{1}{3}t^3 - t$. We see that away from the stationary point of $\frac{1}{3}t^3 - t$, the areas between the t-axis and $\cos\left(x\left(\frac{1}{3}t^3 - t\right)\right)$ approximately cancel each other—which is rigorously embodied in the Riemann-Lebesgue Lemma.

CHAPTER 6

Method of steepest descents

This method originated with Riemann (1892). It is a general technique for finding the asymptotic behaviour of integrals of the form

$$I(\lambda) = \int_{\mathcal{C}} f(z)\, \mathrm{e}^{\lambda h(z)}\, \mathrm{d}z\,,$$

as $\lambda \to +\infty$, where $\mathcal{C}$ is a contour in the complex z-plane and $f(z)$ & $h(z)$ are analytic functions of z in some domain of the complex plane that contains $\mathcal{C}$. The functions $f(z)$ and $h(z)$ need not be analytic in the whole of the complex plane $\mathbb{C}$, in fact frequently in practice, they have isolated singularities, including branch points, the branch lines which must be carefully noted when proceeding with the analysis below (for more details see Murray [**9**] and Bender & Orszag [**2**]).

Suppose $h(z) = \phi(z) + \mathrm{i}\psi(z)$, with $\phi(z)$ and $\psi(z)$ both real valued, and the contour $\mathcal{C}$, which may be finite or infinite, joins the point $z = a$ and $z = b$. Then, with $\mathrm{d}s = |\mathrm{d}z|$, we note that

$$\begin{aligned} |I(\lambda)| &\le \int_{z=a}^{z=b} \left| f(z)\, \mathrm{e}^{\lambda h(z)} \right| \,\mathrm{d}s \\ &\le \int_{z=a}^{z=b} |f(z)|\, \mathrm{e}^{\lambda \phi(z)}\, \mathrm{d}s. \end{aligned}$$

If $\int_{z=a}^{z=b} |f(z)|\, \mathrm{d}s$ is bounded, then by Laplace's method

$$|I(\lambda)| = \mathcal{O}(\mathrm{e}^{\lambda\phi})\,,$$

ignoring multiplicative algebraic terms like $\lambda^{-\frac{1}{2}}$, λ^{-1}, etc.. If $|\mathcal{C}|$ is bounded,

$$|I(\lambda)| \le |\mathcal{C}| \cdot \max_{\mathcal{C}} \left\{ |f(z)|\, \mathrm{e}^{\lambda\phi(z)} \right\}.$$

Hence we observe that the most important contribution to the asymptotic approximation of $|I(\lambda)|$ as $\lambda \to +\infty$ comes from the neighbourhood of the maximum of ϕ.

Basic idea. We can deform the contour $\mathcal{C}$ to a new contour $\mathcal{C}'$, using Cauchy's theorem, at least in the domain of analyticity of $f(z)$ and $h(z)$. If $f(z)$ has an isolated pole singularity for example, we can still deform the contour $\mathcal{C}$ into another, which may involve crossing the singularity provided we use the theory of residues appropriately (branch points/cuts need more delicate care).

The reason for deforming $\mathcal{C} \to \mathcal{C}'$ is twofold:

(1) we want to deform the path $\mathcal{C}$ so that ϕ drops off either side of its maximum at the fastest possible rate. In this way the largest value of $\mathrm{e}^{\lambda\phi(z)}$ as $\lambda \to +\infty$ will be concentrated in a small section of the contour. This specific path through the point of the maximum of $\mathrm{e}^{\lambda\phi(z)}$ will be the *contour of steepest descents*. It is important to note however, that deforming the contour $\mathcal{C}$ in this way may alter its length and drastically change the variation of $\mathrm{e}^{\lambda\phi(z)}$ in the neighbourhood of its maximum when λ is large.

(2) we want to deform the path $\mathcal{C}$ so that $h(z)$ has a constant imaginary part, i.e. so that ψ is constant. The purpose of this is to eliminate the necessity to consider rapid oscillations of the integrand when λ is large. Such a contour is known as a *constant–phase contour* and would mean that

$$I(\lambda) = \mathrm{e}^{\mathrm{i}\lambda\psi} \int_{\mathcal{C}'} f(z)\, \mathrm{e}^{\lambda\phi(z)}\, \mathrm{d}z\,,$$

which although z is complex, can be treated by Laplace's method as $\lambda \to +\infty$ because $\phi(z)$ is real. (Alternatively we could deform $\mathcal{C}$ so that ϕ is constant on it, and apply the method of stationary phase. However, Laplace's method is a much better approximation scheme as a full asymptotic expansion of a Laplace integral is determined by the integrand in an arbitrary small neighbourhood of the maximum of $\phi(z)$ on the contour; whereas the full asymptotic expansion of a generalized Fourier integral depends on the behaviour of the integrand along the entire contour.)

So the question is: can we deform $\mathcal{C} \to \mathcal{C}'$ so that both these goals can be satisfied?

Recall. For a differentiable function $f(x, y)$ of two variables, $\nabla f(x, y)$ points in the direction of the greatest rate of change of f at the point (x, y). The directional derivative in the direction of the unit vector $\boldsymbol{n}$ is $\partial f/\partial n = \nabla f \cdot \boldsymbol{n} =$ the rate of f in the direction $\boldsymbol{n}$. Hence on a two dimensional contour plot, ∇f is perpendicular to level contours of f, and the directional derivative in the direction parallel to the contour is zero.

Let $h(z)$ be analytic in z, and for the moment, let us restrict ourselves to regions of $\mathbb{C}$ where $h'(z) \neq 0$. We define a *constant–phase contour* of $\mathrm{e}^{\lambda h(z)}$, where $\lambda > 0$, as a contour on which $\psi = \operatorname{Re} h(z)$ is constant. We define a *steepest contour* as one whose tangent is parallel to $\nabla|\mathrm{e}^{\lambda h(z)}| = \nabla \mathrm{e}^{\lambda\phi(z)}$, which is parallel to $\nabla\phi(z)$, i.e. a steepest contour is one on which $\mathrm{e}^{\lambda h(z)}$ is changing most rapidly in z.

The important result here is, if $h(z)$ is analytic with $h'(z) \neq 0$, then

constant–phase contours are steepest contours.

To prove this result, we note that since $h(z)$ is analytic, then its real and imaginary parts satisfy the Cauchy-Riemann equations,

$$\phi_x = \psi_y \quad \text{and} \quad \phi_y = -\psi_x.$$

Hence $\nabla\phi \cdot \nabla\psi = 0$; and thus $\nabla\phi$ is perpendicular to $\nabla\psi$ provided $h'(z) \neq 0$. So the directional derivative of ψ in the direction of $\nabla\phi$ is zero, which means that ψ is constant on contours whose tangents are parallel to $\nabla\phi$.

Note that where $h'(z) \neq 0$, there is a unique constant–phase/steepest contour through that point.

Saddle points. When the contour of integration $\mathcal{C}$ is deformed into a constant phase contour $\mathcal{C}'$, we determine the leading order asymptotic behaviour of $I(\lambda)$ from the behaviour of the integrand in the vicinity of the local maxima of $\phi(z)$—the local maxima may occur at an interior point of a contour or an endpoint. If the local maximum occurs at an interior point of the contour, then the directional derivative of ϕ along the contour vanishes, and the Cauchy-Riemann equations imply that $\nabla\phi = \nabla\psi = 0$. Hence $h'(z) = 0$ at an interior maximum of ϕ on a constant–phase contour.

A point where $h'(z) = 0$ is called a *saddle point*; at such points, two or more level contours of ψ, and hence also steepest contours, may intersect. Note that by the maximum modulus theorem, ϕ and ψ, *cannot* have a maximum (or a minimum) in the domain of analyticity of $h(z)$ (where they are harmonic).

General procedure. First note that near the saddle point z_0, we can expand $h(z)$ as a Taylor series

$$h(z) = h(z_0) + \frac{1}{2}h''(z_0) \cdot (z - z_0)^2 + \mathcal{O}\left((z - z_0)^3\right).$$

We now deform the contour $\mathcal{C}$ (assuming in this case that the endpoints lie in the valleys on either side of the saddle point) so that it lies along the steepest descent path obtained by setting $\psi(z) = \psi(z_0)$. On this path, near z_0,

$$\phi(z) - \phi(z_0) = h(z) - h(z_0) + \frac{1}{2}h''(z_0) \cdot (z - z_0)^2 < 0\,,$$

which is real (the constant imaginary parts cancel). Let us introduce the new *real* variable τ by

$$h(z) - h(z_0) = -\tau^2\,, \tag{6.1}$$

which determines z as a function of τ, i.e. $z(\tau)$. Hence

$$I(\lambda) = \mathrm{e}^{\lambda h(z_0)} \int_{\tau_a}^{\tau_b} f(z(\tau))\, \mathrm{e}^{-\lambda\tau^2} \cdot \frac{\mathrm{d}z}{\mathrm{d}\tau}\, \mathrm{d}\tau\,,$$

where $\tau_a > 0$ and $\tau_b > 0$ correspond to the endpoints $z = a$ and $z = b$ of the original contour under the transformation (6.1). Laplace's method then implies that as $\lambda \to +\infty$,

$$I(\lambda) \sim \mathrm{e}^{\lambda h(z_0)} \int_{-\infty}^{+\infty} f(z(\tau))\, \mathrm{e}^{-\lambda\tau^2} \cdot \frac{\mathrm{d}z}{\mathrm{d}\tau}\, \mathrm{d}\tau\,. \tag{6.2}$$

In this last expression, $z(\tau)$, and hence $f(z(\tau))$ & $\mathrm{d}z/\mathrm{d}\tau$, are obtained by inverting the transformation (6.1) on the steepest descent contour with $\psi = \psi(z_0)$: since

$$\frac{1}{2}h''(z_0) \cdot (z - z_0)^2 + \mathcal{O}\left((z - z_0)^3\right) = -\tau^2\,,$$

we get

$$z - z_0 = \sqrt{\frac{-2}{h''(z_0)}} \cdot \tau + \mathcal{O}(\tau^2)\,. \tag{6.3}$$

Now we require $f(z(\tau))$ as a power series

$$\begin{aligned} f(z(\tau)) &= f(z_0) + f'(z_0) \cdot (z - z_0) + \cdots \\ &= f(z_0) + f'(z_0) \cdot \sqrt{\frac{-2}{h''(z_0)}} \cdot \tau + \mathcal{O}(\tau^2)\,. \end{aligned}$$

Substituting our expansions for $f(z(\tau))$ and $\mathrm{d}z/\mathrm{d}\tau$ into (6.2), we get as $\lambda \to +\infty$,

$$I(\lambda) \sim \mathrm{e}^{\lambda h(z_0)} \cdot f(z_0) \cdot \sqrt{\frac{-2}{h''(z_0)}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda\tau^2}\, \mathrm{d}\tau + \cdots\,.$$

$$\Rightarrow I(\lambda) = f(z_0) \cdot \sqrt{\frac{-2\pi}{\lambda h''(z_0)}} \cdot \mathrm{e}^{\lambda h(z_0)} + \mathcal{O}\left(\frac{\mathrm{e}^{\lambda h(z_0)}}{\lambda}\right)\,.$$

Note. It is important in (6.3), since $h''(z_0)$ is complex, to choose the appropriate branch of $\sqrt{\frac{-1}{h''(z_0)}}$ when z lies on the steepest descent path—it must be chosen consistent with the direction of passage through the stationary point. If there is more than one stationary point, then the relevant stationary point is the one which admits a possible deformation of the original contour into a path of steepest descents. If more than one stationary point is appropriate, then the one which gives the maximum ϕ is the relevant one. The method of steepest descents still applies even if the endpoints of the original contour only lie in one valley (rather than in valleys either side of the stationary point)—we still deform the contour to a path of steepest descents. If the stationary point is of order $m-1$, i.e. $h'(z_0) = h''(z_0) = \cdots = h^{(m-1)}(z_0) = 0$ and $h^{(m)}(z_0) \neq 0$, then approximate $h(z)$ near z_0 by $h(z) = h(z_0) + \frac{1}{m!}h^{(m)}(z_0) \cdot (z - z_0)^m$ instead, and proceed as before.

Example. Consider the Gamma function, with a complex argument,

$$\Gamma(a+1) \equiv \int_0^\infty \mathrm{e}^{-t}\, t^a\, \mathrm{d}t\,. \tag{6.4}$$

The path of integration is along the real axis and $a \in \mathbb{C}$, with $|\arg(a)| < \pi/2$. We wish to find the asymptotic expansion of $\Gamma(a+1)$ as $a \to \infty$ along some ray. Since $a \in \mathbb{C}$, the principal value of t^a is taken with the branch line as the negative real t-axis. To apply the method of steepest descents, we must first express (6.4) in the appropriate form; consider the transformation

$$t = a(1+z)\,, \quad a = \lambda \mathrm{e}^{\mathrm{i}\alpha}\,, \quad \lambda > 0\,, \quad |\alpha| < \frac{\pi}{2}\,,$$

where now z is complex, and the contour $\mathcal{C}$ in the z-plane goes from $z=-1$ ($t=0$) to $z=\infty\,\mathrm{e}^{-\mathrm{i}\alpha}$ ($t=+\infty$) with $|\alpha|<\pi/2$:

$$\Gamma(a+1)=a^{a+1}\mathrm{e}^{-a}\int_{-1}^{\infty\,\mathrm{e}^{-\mathrm{i}\alpha}}\mathrm{e}^{\lambda\left(\mathrm{e}^{\mathrm{i}\alpha}(\log(1+z)-z)\right)}\,\mathrm{d}z\,,$$

where the principal value of the logarithm is taken with the branch line from $z=-1$ along the negative real axis to $z=\infty\,\mathrm{e}^{\mathrm{i}\pi}$. Hence we are now interested in the asymptotic approximation as $\lambda\to+\infty$ of

$$I(\lambda)=\int_{\mathcal{C}}\mathrm{e}^{\lambda h(z)}\,\mathrm{d}z\,,$$

where $h(z)=\mathrm{e}^{\mathrm{i}\alpha}\,(\log(1+z)-z)$ and $\mathcal{C}$ is the contour already described.

There is a single stationary point of $h(z)$:

$$h'(z)=\mathrm{e}^{\mathrm{i}\alpha}\left(\frac{1}{1+z}-1\right)=0\quad\Leftrightarrow\quad z=0\,.$$

Since $h''(0)=-1\neq0$, $z=0$ is a stationary point of order 1. The path of steepest descents through $z=0$ is

$$\psi=\mathrm{Im}\,\{h(z)\}=\mathrm{Im}\,\{h(0)\}=0\,,$$

which, when expanding $h(z)$ close to $z=0$ is (setting $z=r\mathrm{e}^{\mathrm{i}\theta}$, $r>0$) approximately

$$\mathrm{Im}\left\{-\mathrm{e}^{\mathrm{i}\alpha}\cdot\frac{1}{2}z^2\right\}\approx0\quad\Rightarrow\quad r^2\sin(2\theta+\alpha)\approx0\,.$$

The paths of steepest descent and ascent near $z=0$ are

$$\theta=-\frac{1}{2}\alpha\quad\text{with the continuation}\quad\theta=\pi-\frac{1}{2}\alpha,\tag{6.5}$$

$$\theta=-\frac{1}{2}\pi-\frac{1}{2}\alpha\quad\text{with the continuation}\quad\theta=\frac{1}{2}\pi-\frac{1}{2}\alpha\,.\tag{6.6}$$

Along (6.5) near $z=0$,

$$\phi=\mathrm{Re}\,\{h(z)\}\approx\mathrm{Re}\left\{-\mathrm{e}^{\mathrm{i}\alpha}\cdot\frac{1}{2}z^2\right\}=-\frac{1}{2}r^2\cos(2\theta+\alpha)<0=\phi(0)$$

and so (6.5) must be the path of steepest descents while (6.6) is the path of steepest ascents.

We now deform $\mathcal{C}$ to the steepest descents path in the vicinity of the stationary point, $\mathcal{C}'$, shown in the figure. Following the general procedure, introduce the new real variable τ by,

$$h(z)-h(0)=\mathrm{e}^{\mathrm{i}\alpha}\,(\log(1+z)-z)=-\tau^2\,.$$

Expanding the left-hand side in Taylor series near the stationary point $z=0$ gives

$$-\mathrm{e}^{\mathrm{i}\alpha}\cdot\frac{1}{2}z^2+\cdots=-\tau^2\quad\Rightarrow\quad z(\tau)=\pm\sqrt{2}\cdot\mathrm{e}^{-\mathrm{i}\alpha/2}\cdot\tau+\mathcal{O}(\tau^2)\,,$$

where ± 1 are the two branches of $\sqrt{1}$. On the steepest descents path $\theta = -\alpha/2$ near $z = 0$; we wish to have $\tau > 0$. Since $z = r\mathrm{e}^{-\mathrm{i}\alpha/2}$ on it, we must choose the plus sign so

$$z(\tau) = +\sqrt{2} \cdot \mathrm{e}^{-\mathrm{i}\alpha/2} \cdot \tau + \mathcal{O}(\tau^2)\,.$$

Hence as $\lambda \to +\infty$,

$$I(\lambda) = \sqrt{2} \cdot \mathrm{e}^{-\mathrm{i}\alpha/2} \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda\tau^2}\, \mathrm{d}\tau = \sqrt{\frac{2\pi}{\lambda}} \cdot \mathrm{e}^{-\mathrm{i}\alpha/2} = \sqrt{\frac{2\pi}{a}}\,.$$

And so

$$\Gamma(a+1) = \sqrt{2\pi} \cdot a^{a+\frac{1}{2}} \mathrm{e}^{-a} \quad \text{as} \quad a \to \infty \quad \text{with} \quad |\arg(a)| < \frac{\pi}{2}\,.$$

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APPENDIX A

Notes

A.1. Remainder theorem

THEOREM. *(Remainders) Suppose* $f^{(n-1)}$ *is continuous in* $[a, a+h]$ *and differentiable in* $(a, a+h)$. *Let* R_n *be defined by*

$$f(a+h) = \sum_{m=0}^{n-1} \frac{f^{(m)}(a)}{m!} \cdot h^m + R_n.$$

Then

(1) *(with Legendre's remainder) there exists* $\theta \in (0,1)$ *such that*

$$R_n = \frac{f^{(n)}(a+\theta h)}{n!} \cdot h^n;$$

(2) *(with Cauchy's remainder) there exists* $\tilde{\theta} \in (0,1)$ *such that*

$$R_n = (1-\tilde{\theta})^{n-1} \frac{f^{(n)}(a+\tilde{\theta} h)}{(n-1)!} \cdot h^n;$$

In particular, if we can prove that $R_n \to 0$ *as* $n \to \infty$ *(in either form above), then the* Taylor expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot h^n$$

converges to $f(a+h)$.

A.2. Taylor series for functions of more than one variable

We can successively attempt to approximate $f(x,y)$ near $(x,y) = (a,b)$ by a tangent plane, quadratic surface, cubic surface, etc..., to generate the Taylor series expansion for $f(x,y)$:

$$\begin{aligned} f(x,y) = f(a,b) &+ f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+ \frac{1}{2!}\Big(f_{xx}(a,b)(x-a)^2 \\ &+ 2f_{x,y}(a,b)(x-a)(y-b) \\ &+ f_{yy}(a,b)(y-b)^2\Big) + \cdots . \end{aligned} \tag{A.1}$$

Note that we can also derive the above series by expanding

$$F(t) = f(a+ht, b+kt) \tag{A.2}$$

in a Taylor series in the single independent variable t, about $t = 0$, with $h = x - a$, $k = y - b$, and evaluating at $t = 1$:

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F''(0) + \cdots . \tag{A.3}$$

Now using that $F(t)$ is given by (A.2) in (A.3), we can generate (A.1).

A.3. How to determine the expansion sequence

Example. Consider the perturbation problem

$$(1-\epsilon)x^2 - 2x + 1 = 0\,, \tag{A.4}$$

for small ϵ. When $\epsilon = 0$ this quadratic equation has a double root

$$x_0 = 1\,.$$

For small $\epsilon \neq 0$ the quadratic equation (A.4) will have two distinct roots, both close to 1. To determine the appropriate expansion for both roots, we pose a general expansion of the form

$$x(\epsilon) = 1 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + \cdots\,, \tag{A.5}$$

where we require that

$$1 \gg \delta_1(\epsilon) \gg \delta_2(\epsilon) \gg \cdots\,, \tag{A.6}$$

and that x_1, x_2, ... are *strictly order unity* as $\epsilon \to 0$ (this means that they are $\mathcal{O}(1)$, but not asymptotically small as $\epsilon \to 0$).

The idea is then to substitute (A.5) into (A.4) and, after cancelling off any obvious leading terms and noting the asymptotic relations (A.6) and $|\epsilon| \ll 1$, to consider the three possible leading order balances: $\delta_1^2 \gg \epsilon$, $\delta_1^2 = \epsilon$ and $\delta_1^2 \ll \epsilon$. For more details see Hinch [**5**].

A.4. How to find a suitable rescaling

For some singular perturbation problems it is difficult to guess what the appropriate rescaling should be a-priori. In this case, pose a general rescaling factor $\delta(\epsilon)$:

$$x = \delta X\,, \tag{A.7}$$

where we will insist that X is *strictly of order unity* as $\epsilon \to 0$. Then substitute the rescaling (A.7) into the singular perturbation problem at hand, and examine the balances within the rescaled problem for δ of different magnitudes.

Example. Consider the singular perturbation problem

$$\epsilon x^3 + x^2 - 1 = 0\,, \tag{A.8}$$

for small ϵ. In the limit as $\epsilon \to 0$, one of the roots is knocked out while the other two remaining roots are

$$x_0 = \pm 1\,.$$

For small $\epsilon \neq 0$ the cubic equation will have two regular roots close to ± 1; approximate expansions for these roots in terms of powers of ϵ can be found in the usual way. To determine the appropriate expansion for the remaining singular root we substitute the rescaling (A.7) into (A.8) to get

$$\epsilon\delta^3 X^3 + \delta^2 X^2 - 1 = 0\,. \tag{A.9}$$

It is quite clear in this simple example that the appropriate rescaling that regularizes the problem is $\delta = 1/\epsilon$—we could also use the argument that we expect the singular root to be large for small ϵ and hence at leading order we have

$$\epsilon x^3 + x^2 \approx 0 \qquad \Rightarrow \qquad x \approx -\frac{1}{\epsilon}\,.$$

However let's suppose that we did not see this straight away and try to deduce it by considering δ of different magnitudes and examining the relative size of the terms in the left-hand side of (A.9) as follows.

- If $\delta \ll 1$, then
$$\underbrace{\epsilon\delta^3 X^3}_{\mathrm{o}(1)} + \underbrace{\delta^2 X^2}_{\mathrm{o}(1)} - 1\,.$$
where by o(1) we of course mean asymptotically small ($\ll 1$). With this rescaling the left-hand side of (A.9) clearly cannot balance zero on the right-hand side and so this rescaling is unacceptable.

- If $\delta = 1$, then
$$\underbrace{\epsilon\delta^3 X^3}_{\mathrm{o}(1)} + \delta^2 X^2 - 1\,.$$
We can balance zero (on the right-hand side) in this case and this rescaling corresponds to the trivial one we would make for the two regular roots, i.e. we get that $X = \pm 1 + \mathrm{o}(1)$.

- If $1 \ll \delta \ll \epsilon^{-1}$, then (after dividing through by δ^2)
$$\underbrace{\epsilon\delta X^3}_{\mathrm{o}(1)} + X^2 - \underbrace{1/\delta^2}_{\mathrm{o}(1)}\,.$$
For this rescaling we cannot balance zero on the right-hand side unless $X = 0 + \mathrm{o}(1)$, but this violates our assumption that X is strictly order unity. Hence this rescaling in unacceptable.

- If $\delta = \epsilon^{-1}$, then (after dividing through by $\epsilon\delta^3$)
$$X^3 + X^2 - \underbrace{1/\epsilon\delta^3}_{\mathrm{o}(1)}\,.$$
This can balance zero on the right-hand side if
$$X = -1 + \mathrm{o}(1)\,,$$
or
$$X = 0 + \mathrm{o}(1)\,.$$
The first solution must correspond to the singular root, whilst the second violates our assumption that X is strictly order unity.

- If $\epsilon^{-1} \ll \delta$, then (after dividing through by $\epsilon\delta^3$)

$$X^3 + \underbrace{\tfrac{1}{\epsilon\delta}\,X^2}_{\mathrm{o}(1)} - \underbrace{1/\epsilon\delta^3}_{\mathrm{o}(1)}\ .$$

  This cannot balance zero on the right-hand side unless $X = 0 + \mathrm{o}(1)$, but this violates our assumption that X is strictly order unity. Hence this rescaling in unacceptable.

Hence $\delta = 1$ is the suitable rescaling for the *regular root* and $\delta = \epsilon^{-1}$ is the suitable rescaling for investigating the *singular root*.

APPENDIX B

Exam formula sheet

Power series.

$$\mathrm{e}^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \quad \text{for all} \quad x$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots \quad \text{for all} \quad x$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots \quad \text{for all} \quad x$$

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots \quad \text{for all} \quad x$$

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \quad \text{for all} \quad x$$

$$(a+x)^k = a^k + ka^{k-1}x + \frac{k(k-1)}{2!}a^{k-2}x^2 + \cdots \quad \text{for} \quad |x| < a$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \pm \cdots \quad \text{for} \quad |x| < 1$$

Integration by parts formula.

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = \big[u(t)v(t)\big]_{t=a}^{t=b} - \int_a^b u'(t)v(t)\,\mathrm{d}t\,.$$

Perturbation expansions.

$$x(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + x_4\epsilon^4 + \mathcal{O}(\epsilon^5)$$

$$\begin{aligned}\big(x(\epsilon)\big)^2 &= x_0^2 + 2x_0x_1\epsilon + (x_1^2 + 2x_0x_2)\epsilon^2 + (2x_1x_2 + 2x_0x_3)\epsilon^3 \\ &\quad + (x_2^2 + 2x_1x_3 + 2x_0x_4)\epsilon^4 + \mathcal{O}(\epsilon^5)\end{aligned}$$

$$\begin{aligned}\big(x(\epsilon)\big)^3 &= x_0^3 + 3x_0^2x_1\epsilon + 3(x_0x_1^2 + x_0^2x_2)\epsilon^2 + (x_1^3 + 6x_0x_1x_2 + 3x_0^2x_3)\epsilon^3 \\ &\quad + 3(x_1^2x_2 + 2x_0x_1x_3 + x_0x_2^2 + x_0^2x_4)\epsilon^4 + \mathcal{O}(\epsilon^5)\end{aligned}$$

$$\begin{aligned}\big(x(\epsilon)\big)^4 &= x_0^4 + 4x_0^3x_1\epsilon + 2(3x_0^2x_1^2 + 2x_0^3x_2)\epsilon^2 + 4(x_0x_1^3 + 3x_0^2x_1x_2 + x_0^3x_3)\epsilon^3 \\ &\quad + (x_1^4 + 12x_0x_1^2x_2 + 6x_0^2x_2^2 + 12x_0^2x_1x_3 + 4x_0^3x_4)\epsilon^4 + \mathcal{O}(\epsilon^5)\end{aligned}$$

Definite integrals.

$$\int_0^\infty s^{n-1}\,\mathrm{e}^{-s}\,\mathrm{d}s = n! \quad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

$$\int_0^\infty s^{\alpha-1}\,\mathrm{e}^{-s}\,\mathrm{d}s = \Gamma(\alpha) \quad \text{for} \quad \alpha > 0$$

$$\int_{-\infty}^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}$$

$$\int_{-\infty}^\infty s^2\,\mathrm{e}^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^\infty \mathrm{e}^{\pm\mathrm{i}s^2}\,\mathrm{d}s = \sqrt{\pi}\,\mathrm{e}^{\pm\mathrm{i}\pi/4}$$