...approach suggested by

Arieh Iserles
Outline of the talk

1. Some (well known) Lie group methods for linear problems (Fer and Magnus expansions).

2. Schemes based on triangular matrices (splitting + solvability).

3. Some methods and practical issues in their construction
Let us consider a linear matrix ODE evolving in a Lie group $\mathcal{G}$

\[ Y' = A(t)Y, \quad Y(t_0) = Y_0 \in \mathcal{G} \]  

(0)

with $A : [t_0, \infty] \times \mathcal{G} \rightarrow \mathfrak{g}$ smooth enough.

$\mathfrak{g}$: Lie algebra associated with $\mathcal{G}$

Examples of $\mathcal{G}$: $\text{SL}(n)$, $\text{O}(n)$, $\text{SU}(n)$, $\text{Sp}(n)$, $\text{SO}(n)$, ...

$Y(t) \in$ Lie group $\mathcal{G}$ if $A(t) \in$ Lie algebra $\mathfrak{g}$

* There are several schemes preserving this feature (Magnus, Fer, Cayley,...)
1.1 Magnus expansion

For the equation

\[ Y' = A(t)Y, \quad Y(t_0) = I, \]

* Magnus (1954) proposed

\[ Y(t) = e^{\Omega(t)}, \quad \Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t) \]
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\[ Y(t) = e^{\Omega(t)}, \quad \Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t) \tag{1} \]

with \( \log(Y(t)) \) satisfying

\[ \Omega' = d \exp_{\Omega}^{-1} A(t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \text{ad}^k_{\Omega} A(t), \quad \Omega(t_0) = 0, \tag{2} \]
Here

\[
\begin{align*}
\text{ad}^0_\Omega A & = A \\
\text{ad}^k_\Omega A & = [\Omega, \text{ad}^{k-1}_\Omega A] \\
[\Omega, A] & \equiv \Omega A - A\Omega
\end{align*}
\]

and \( B_k \) are Bernoulli numbers.
First terms in the expansion \((A_i \equiv A(t_i))\):

\[
\begin{align*}
\Omega_1(t) & = \int_{t_0}^{t} A(t_1) dt_1 \\
\Omega_2(t) & = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [A_1, A_2] \\
\Omega_3(t) & = \frac{1}{6} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 ([A_1, [A_2, A_3]] + [A_3, [A_2, A_1]])
\end{align*}
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\(e^{\Omega(t)} \in \mathcal{G}\) even if the series \(\Omega\) is truncated
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\( e^{\Omega(t)} \in G \) even if the series \( \Omega \) is truncated

* Expansion widely used in Quantum Mechanics, NMR spectroscopy, infrared divergences in QED, control theory,...
1.1 Magnus expansion (IV)

Magnus as a numerical integration method (Iserles & Nørsett, 1997)
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Two critical factors in the computational cost of the resulting algorithms:

1. Evaluation of $\exp(\Omega)$
   
   (Moler & Van Loan, Celledoni & Iserles, ...)

2. Number of commutators involved in the expansion

To reduce this number is particularly useful the concept of **graded free Lie algebra** (Munthe-Kaas, Owren 1999)
1.1 Magnus expansion (V)

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* Numerical schemes based on Magnus up to order 8 have been constructed involving the minimum number of commutators in terms of quadratures and/or univariate integrals.

* Efficient in applications
1.2 Other schemes

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1.2 Other schemes

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* Referred erroneously in the (mathematical physics) literature (e.g., Wilcox 1967), but ...

* proposed (as an exercise!) by R. Bellman, ‘Introduction to Matrix Analysis’, 1960, page 204:

“The solution of \( dX/dt = Q(t)X \), \( X(0) = I \), can be put in the form 
\[ e^P e^{P_1} \cdots e^{P_n} \cdots, \] 
where \( P = \int_0^t Q(s)ds \), and \( P_n = \int_0^t Q_n ds \), with

\[
Q_n = e^{-P_{n-1}} Q_{n-1} e^{P_{n-1}} + \int_0^{-1} e^{sP_{n-1}} Q_{n-1} e^{-sP_{n-1}} ds
\]

The infinite product converges if \( t \) is sufficiently small.”

(See also Mathematical Reviews 21 2771, review done by R. Bellman)
1.2 Other schemes (II)

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* This class of methods can actually be built from Magnus.
* They require the computation of several matrix exponentials.
Let us suppose that $Y' = A(t) Y$ is defined in a $J$-orthogonal Lie group,

$$O_J(n) = \{ A \in \text{GL}_n(\mathbb{R}) : A^T J A = J \},$$

$J$: constant matrix
1.3 Methods based on the Cayley transform

Let us suppose that $Y' = A(t)Y$ is defined in a $J$-orthogonal Lie group,

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$J$: constant matrix

Examples: orthogonal group ($J = I$), symplectic group, Lorentz group ($J = \text{diag}(1, -1, -1, -1)$).

Solution:

$$Y(t) = \left( I - \frac{1}{2}C(t) \right)^{-1} \left( I + \frac{1}{2}C(t) \right)$$
1.3 Methods based on the Cayley transform (II)

with $C(t) \in \mathfrak{o}_J(n)$ satisfying (Iserles 2001)

$$\frac{dC}{dt} = A - \frac{1}{2} [C, A] - \frac{1}{4} CAC, \quad t \geq t_0, \quad C(t_0) = 0.$$ 

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* It is possible to construct methods which are more efficient than those based on the Cayley transform (Blanes, C., Ros 2002).
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1.4 Summary

* These methods require the evaluation of one or several matrix exponentials

⇒ They are expensive when $n$ is very large

* In some cases, if the exponential is approximated by rational functions the method does not preserve the Lie group structure, in particular, when $\mathcal{G} = \text{SL}(n)$

⇒⇒ Another class of methods is required.
The procedure

For the linear system

\[ Y' = A(t)Y, \quad Y(0) = I, \]

we denote \( Y_0 \equiv Y, A_0 \equiv A \) and suppose that

\[ A_0(t) = A_{0+}(t) + A_{0-}(t), \]

where

\( A_{0+} \in \nabla_n \) is strictly upper-triangular

\( A_{0-} \in \tilde{\Delta}_n \) is weakly lower-triangular.
2 Solvability + splitting (II)

The idea is to write the solution as a product of upper and lower triangular matrices.
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More specifically, we propose the following factorization:

$$Y_0(t) = L_0(t)U_0(t)Y_1(t)$$

such that

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such that

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Observe then that \( L_0(t) \) can be obtained by quadratures and \( L_0(t) \in \tilde{\Delta}_n \).
Now we form the matrix

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which can also be split as

\[ C_0(t) = C_{0+}(t) + C_{0-}(t), \]

where

\[ C_{0+} \in \nabla_n \text{ is weakly upper-triangular} \]

\[ C_{0-} \in \Delta_n \text{ is strictly lower-triangular}. \]
Next we choose $U_0$ as the solution of

$$U'_0 = C_0(t)U_0, \quad U_0(0) = I$$

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It is easy to show that $Y_1$ satisfies

$$Y'_1 = A_1(t)Y_1, \quad Y_1(0) = I,$$

with

$$A_1 = U_0^{-1}C_{0_-}U_0.$$
This gives a single step of the solvable cycle, which we repeat with $A_1$.

$$A_1 = A_{1_+} + A_{1_-}, \quad A_{1_+} \in \bigtriangledown_n, \quad A_{1_-} \in \tilde{\bigtriangleup}_n$$

$$Y_1 = L_1 U_1 Y_2$$

$$L'_1 = A_{1_-} L_1, \quad L_1(0) = I$$

e tc.
In this way one has the following algorithm:

\[ Y \equiv Y_0 = L_0 U_0 L_1 U_1 \cdots L_k U_k Y_{k+1} \]

with \((k = 0, 1, 2, \ldots)\)

\[ A_k = A_{k+} + A_{k-}, \quad A_{k+} \in \bigtriangledown_n, \quad A_{k-} \in \tilde{\bigtriangleup}_n \]

\[ L'_{k} = A_{k-} L_k, \quad L_k(0) = I \]

\[ C_k \equiv L_k^{-1} A_{k+} L_k = C_{k+} + C_{k-} \]

\[ C_{k+} \in \tilde{\bigtriangledown}_n, \quad C_{k-} \in \bigtriangleup_n \]

\[ U'_{k} = C_{k+} U_k, \quad U_k(0) = I \]
and finally

\[ A_{k+1} \equiv U_k^{-1} C_k U_k, \quad Y'_{k+1} = A_{k+1} Y_{k+1} \]
2 Solvability + splitting (VII)

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Usually the factorization is truncated by taking \( Y_{k+1} = I \).
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In what follows we will analyse the main features of this procedure as a numerical integrator.
2.1 Order of the method

Suppose that \( A(t) = \varepsilon(a_0 + a_1 t + a_2 t^2 + \cdots) \) for some parameter \( \varepsilon > 0 \).
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Suppose that $A(t) = \varepsilon (a_0 + a_1 t + a_2 t^2 + \cdots)$ for some parameter $\varepsilon > 0$. Then

\[
A_{j-} = t^{n_j} \varepsilon^{n_j} (\varepsilon \alpha_1 + t(\varepsilon \alpha_2 + \varepsilon^2 \alpha_3) + O(t^2))
\]
\[
A_{j+} = t^{m_j} \varepsilon^{m_j} (\varepsilon \beta_1 + t(\varepsilon \beta_2 + \varepsilon^2 \beta_3) + O(t^2))
\]

for $j = 1, 2, \ldots$, so that

\[
L_j(t) = I + \frac{1}{n_j + 1} (t\varepsilon)^{n_j+1} \alpha_1 + \frac{1}{n_j + 2} t^{n_j+2} \varepsilon^{n_j} (\varepsilon \alpha_2 + \varepsilon^2 \alpha_3) + \cdots
\]
\[
U_j(t) = I + \frac{1}{m_j + 1} (t\varepsilon)^{m_j+1} \beta_1 + \frac{1}{m_j + 2} t^{m_j+2} \varepsilon^{m_j} (\varepsilon \beta_2 + \varepsilon^2 \beta_3) + \cdots
\]
Furthermore,

\[ n_{j+1} = n_j + m_j + 1 \]

\[ m_{j+1} = n_j + 2m_j + 2 \quad j = 1, 2, \ldots \]
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<table>
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<th>( j )</th>
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(1) This algorithm could be useful for problems of the form

\[ Y' = (B_0 + \epsilon B_1)Y \]

if the system \( Y' = B_0Y \) can be solved exactly.
2.1 Order of the method (III)

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(1) This algorithm could be useful for problems of the form

\[ Y' = (B_0 + \varepsilon B_1)Y \]

if the system \( Y' = B_0Y \) can be solved exactly.

(2) The order of approximation is...
2.1 Order of the method (IV)

\[
\begin{align*}
Y_0 & \approx L_0 U_0 & \text{is order } & 1 \\
Y_0 & \approx L_0 U_0 L_1 & & 2 \\
Y_0 & \approx L_0 U_0 L_1 U_1 & & 4 \\
Y_0 & \approx L_0 U_0 L_1 U_1 L_2 & & 7 \\
Y_0 & \approx L_0 U_0 L_1 U_1 L_2 U_2 & & 12 \\
Y_0 & \approx L_0 U_0 L_1 U_1 L_2 U_2 L_3 & & 20 \\
Y_0 & \approx L_0 U_0 L_1 U_1 L_2 U_2 L_3 U_3 & & 33
\end{align*}
\]
Section 2.1 Order of the method (IV)

With only 4 solvable cycles we get order 33!

\[
Y_0 \approx L_0 U_0 \quad \text{is order} \quad 1
\]
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Y_0 \approx L_0 U_0 L_1 \quad 2
\]
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is order 1

is order 2

is order 4

is order 7

is order 12

is order 20

is order 33

With only 4 solvable cycles we get order 33!

...if we can compute \( L_k \) and \( U_k \) up to this order...
2.2 Questions

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(1) Does the approximate solution evolve in the Lie group if $A$ is in the Lie algebra, i.e., is it a Lie group method?

(2) Solve explicitly the systems $L'_k = A_k - L_k$ and $U'_k = C_k + U_k$
2.2 Questions

Several problems involved

(1) Does the approximate solution evolve in the Lie group if $A$ is in the Lie algebra, i.e., is it a Lie group method?

(2) Solve explicitly the systems $L_k' = A_k L_k$ and $U_k' = C_k + U_k$

(3) Approximate efficiently the (multiple) integrals involved.
3 Practical issues

(1) Preservation of the Lie-group structure

If $A(t) \in \mathfrak{sl}(n)$, the algorithm provides by construction approximations to $Y(t)$ in $\text{SL}(n)$. 
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**Proof.** $A_k = A_{k+} + A_{k-}$, with $A_{k+} \in \nabla_n, A_{k-} \in \tilde{\Delta}_n$. In fact $A_{k-}$ belongs to a solvable subalgebra of $\mathfrak{sl}(n)$. Therefore the solution of

$$L'_k = A_{k-} L_k, \quad L_k(0) = I$$

$L_k(t) \in \text{SL}(n)$ (in fact, a solvable subgroup of).
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$L_k(t) \in \text{SL}(n)$ (in fact, a solvable subgroup of).

$\text{Tr}(A_{k+}) = 0$, and the trace is invariant under similarity, so that

$$\text{Tr}(C_k) = \text{Tr}(L_k^{-1} A_{k+} L_k) = \text{Tr}(A_{k+}) = 0 \Rightarrow C_k \in \mathfrak{sl}(n)$$
Next, $C_k = C_{k_+} + C_{k_-}$, with $C_{k_+} \in \tilde{\nabla}_n$, $C_{k_-} \in \triangle_n$ and $U_k$, solution of

$$U'_k = C_{k_+} U_k, \quad U_k(0) = I$$

belongs to $\mathrm{SL}(n)$. Finally

$$A_{k+1} \equiv U_k^{-1} C_{k_-} U_k \in \mathfrak{sl}(n)$$

and the process is repeated.
Next, $C_k = C_{k+} + C_{k-}$, with $C_{k+} \in \tilde{\nabla}_n$, $C_{k-} \in \triangle_n$ and $U_k$, solution of

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$$A_{k+1} \equiv U_k^{-1} C_{k-} U_k \in \mathfrak{sl}(n)$$

and the process is repeated. □

Other properties (i.e., orthogonality) are preserved only up to the order of the method.
3 Practical issues (III)

(2a) Explicit solution of $L'_k = A_k L_k$
3 Practical issues (III)

(2a) Explicit solution of \( L'_k = A_k L_k \)

Consider \( k = 0 \) and denote \( A_0(t) = (a_{ij}), i, j = 1, \ldots, n, L_0(t) = (L_{ij}), j \leq i \)

\[
A_{ii}(t) \equiv \int_0^t a_{ii}(t_1) dt_1.
\]

Then the solution of \( L'_0 = A_0(t)L_0, L_0(0) = I \) is

\[
L_{ii}(t) = e^{A_{ii}(t)}, \quad i = 1, \ldots, n \tag{3}
\]

\[
L_{ij}(t) = e^{A_{ii}(t)} \int_0^t e^{-A_{ii}(t_1)} \left( \sum_{k=j}^{i-1} a_{ik}(t_1)L_{kj}(t_1) \right) dt_1
\]

\( i = 2, \ldots, n, j = 1, \ldots, i - 1. \)
3 Practical issues (IV)

(2b) Explicit solution of $U'_k = C_k U_k$
3 Practical issues (IV)

(2b) Explicit solution of \( U'_k = C_{k+} U_k \)

Consider \( k = 0 \) and denote \( C_0(t) = (c_{ij}), i, j = 1, \ldots, n, U_0(t) = (U_{ij}), j \geq i \)

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C_{ii}(t) \equiv \int_0^t c_{ii}(t_1) dt_1.
\]

Then the solution of \( U'_0 = C_{0+}(t)U_0, U_0(0) = I \) is

\[
U_{ii}(t) = e^{C_{ii}(t)}, \quad i = 1, \ldots, n
\]

\[
U_{ij}(t) = e^{C_{ii}(t)} \int_0^t e^{-C_{ii}(t_1)} \left( \sum_{k=i+1}^{j} c_{ik}(t_1)U_{kj}(t_1) \right) dt_1
\]

\( i = 1, \ldots, n - 1, j = i + 1, \ldots, n. \)
3 Practical issues (V)

⇒ Explicit expressions for the elements of $L_k$ and $U_k$ in terms of multivariate integrals.
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They can be evaluated in sequence as follows:

\[
\begin{align*}
L_{ii} & : i = 1, \ldots, n \\
L_{i,i-1} & : i = 2, \ldots, n \\
L_{i,i-2} & : i = 3, \ldots, n \\
& \vdots \\
L_{n1} & \\
U_{ii} & : i = 1, \ldots, n \\
U_{i,i+1} & : i = 1, \ldots, n-1 \\
U_{i,i+2} & : i = 1, \ldots, n-2 \\
& \vdots \\
U_{1n} &
\end{align*}
\]
3 Practical issues (VI)

In principle, the integrals appearing in $L_k$ and $U_k$ can be approximated by quadrature rules.
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To minimise the computational cost this has to be done by using the minimum number of $A$ evaluations in each integration step.
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**Question**: Is it possible to approximate *all* the nested integrals with the evaluations required to compute

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i.e., *à la* Magnus?

**YES!**
3.1 Example

Illustration: method of order 4 with 2A evaluations

Step $t = 0 \rightarrow t = h$. 
Illustration: method of order 4 with 2 \( A \) evaluations

Step \( t = 0 \mapsto t = h \).

1- Approximate \( A_{ii}(h), i = 1, \ldots, n \) up to order 4

\[
A_{ii}(h) = \int_0^h a_{ii}(t)dt = \frac{h}{3} (a_{ii}(0) + 4a_{ii}(h/2) + a_{ii}(h)) + O(h^5)
\]

\[
\equiv \tilde{A}_{ii}(h) + O(h^5)
\]

and \( A_{ii}(h/2), i = 1, \ldots, n - 1 \), up to order 3 (necessary to approximate \( L_{ij} \)):

\[
A_{ii}(h/2) = \frac{h}{24} (5a_{ii}(0) + 8a_{ii}(h/2) - a_{ii}(h)) + O(h^4)
\]
3.1 Example (II)

2- $L_{ii}(h) = \exp(\tilde{A}_{ii}(h)) + O(h^5)$ ($i = 1, \ldots, n$) and 
$L_{ii}(h/2) = \exp(\tilde{A}_{ii}(h/2)) + O(h^4)$ ($i = 1, \ldots, n - 1$).

3- Obtain an approximation to $L_{ij}(h)$, $j < i$, of order 4 and $L_{ij}(h/2)$ of order 3

$$L_{ij}(h) = e^{A_{ii}(h)} \int_0^h F_{ij}(t) dt$$

with

$$F_{ij}(t) \equiv e^{-A_{ii}(t)} \sum_{k=j}^{i-1} a_{ik}(t)L_{kj}(t)$$
Then

\[ L_{ij}(h) = e^{\tilde{A}_{ii}(h)} \frac{h}{3} (F_{ij}(0) + 4F_{ij}(h/2) + F_{ij}(h)) + O(h^5) \]

where \( F_{ij}(0) = a_{ij}(0) \) and \( F_{ij}(h/2) \) and \( F_{ij}(h) \) have to be approximated up to order \( h^3 \).

The sequence of computation is \( (i = 2, \ldots, n) \):

(a) \( F_{i,i-1}(h/2) = e^{-\tilde{A}_{ii}(h/2)}a_{i,i-1}(h/2)L_{i-1,i-1}(h/2) + O(h^4) \)

(b) \( F_{i,i-1}(h) = e^{-\tilde{A}_{ii}(h/2)}a_{i,i-1}(h)L_{i-1,i-1}(h) + O(h^5) \)

(c) \( L_{i,i-1}(h), i = 2, \ldots, n \) up to order 4
3.1 Example (IV)

(d) 
\[
L_{i,i-1}(h/2) = e^{\tilde{A}_{ii}(h/2)} \frac{h}{24} (5a_{i,i-1}(0) + 8F_{i,i-1}(h/2) - F_{i,i-1}(h)) + O(h^4)
\]

(e) \(L_{i,i-2}(h), i = 3, \ldots, n\), up to order 4 and \(L_{i,i-2}(h/2)\) up to order 3

...and so on.
3.1 Example (IV)

(d) 
\[ L_{i,i-1}(h/2) = e^{\tilde{A}_{ii}(h/2)} \frac{h}{24} (5a_{i,i-1}(0) + 8F_{i,i-1}(h/2) - F_{i,i-1}(h)) + O(h^4) \]

(e) \[ L_{i,i-2}(h), i = 3, \ldots, n, \text{ up to order } 4 \text{ and } L_{i,i-2}(h/2) \text{ up to order } 3 \]

...and so on.

In this way we have \( L_0(h) \) computed up to order \( O(h^5) \) and also \( L_0(h/2) \) up to order \( O(h^4) \) with 2 evaluations of \( A(t) \).
3.1 Example (V)

3- Next we compute $C_0$:

$$C_0(0) = A_{0+}(0) \quad \text{error } O(h^5)$$

$$C_0(h/2) = L_0^{-1}(h/2)A_{0+}(h/2)L_0(h/2) \quad \text{error } O(h^4)$$

$$C_0(h) = L_0^{-1}(h)A_{0+}(h)L_0(h) \quad \text{error } O(h^5)$$

4- $C_{ii}(h) = \frac{h}{3} \left(c_{ii}(0) + 4c_{ii}(h/2) + c_{ii}(h)\right) + O(h^5)$

$$C_{ii}(h/2) = \frac{h}{24} \left(5c_{ii}(0) + 8c_{ii}(h/2) - c_{ii}(h)\right) + O(h^4)$$
3.1 Example (VI)

5- $U_{i,i+1}(h)$, $i = 1, \ldots, n - 1$, up to order $O(h^5)$;

$U_{i,i+1}(h/2)$, $i = 1, \ldots, n - 1$, up to order $O(h^4)$;

$U_{i,i+2}(h)$, $i = 1, \ldots, n - 2$, up to order $O(h^5)$;

$U_{i,i+2}(h)$, $i = 1, \ldots, n - 2$, up to order $O(h^4)$;

... and so on.

Thus we compute $U_0(h)$ with error $O(h^5)$ and also $U_0(h/2)$ with error $O(h^4)$. 
3.1 Example (VII)

6- $A_1$:

$$
A_1(0) = C_0(0) \quad \text{error } O(h^5)
$$

$$
A_1(h/2) = U_0^{-1}(h/2)C_0(h/2)U_0(h/2) \quad \text{error } O(h^4)
$$

$$
A_1(h) = U_0^{-1}(h)C_0(h)U_0(h) \quad \text{error } O(h^5)
$$

... and the process is repeated again for the second cycle
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... and the process is repeated again for the second cycle

$\Rightarrow$ it is possible to construct a method of order 4 with only 2 $A(t)$ evaluations (3 for the first step).
3.2 Other possibilities

One could use other quadrature rules instead, for instance Gauss–Legendre, but...
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Solution: use G–L with matrix evaluations in the previous/next step.

⇒ method of order 4 with 2 evaluations (and 1 in the next step)
3.3 Some methods

Order 4

\[ Y \approx L_0 U_0 L_1 U_1 \]

* Quadratures NC / GL, 2 matrix evaluations per step

Order 6

\[ Y \approx L_0 U_0 L_1 U_1 L_2 \]

* order 6 with a 5 points NC quadrature (4 evaluations per step)
* order 7 with a 7 points NC (6 evaluations)

Order 12

\[ Y \approx L_0 U_0 L_1 U_1 L_2 U_2 \]

* with a 11 points NC (or GL involving several steps).
3.4 Variable step size

Local extrapolation technique is trivial to implement in this setting.
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Local extrapolation technique is trivial to implement in this setting. For instance,

\[ Y_1 \equiv L_0 U_0 L_1 \]
\[ \hat{Y}_1 \equiv L_0 U_0 L_1 U_1 = Y_1 U_1 \]

Then

\[ \hat{Y}_1 - Y_1 = Y_1 U_1 - Y_1 = Y_1 (U_1 - I) \]

and \( \| \hat{Y}_1 - Y_1 \| \) can be used as a measure of the error.
3.5 Future work

* Analyse the convergence of the procedure
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* Consider numerical examples in $SL(n)$ with (very) large $n$
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* Analyse the convergence of the procedure
* Consider numerical examples in $\text{SL}(n)$ with (very) large $n$
* Highly oscillatory problems (with special quadratures)
* Analyse in practice the preservation of other structures (Blanes & Moan)
* Try to generalize to nonlinear problems
The End

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