Noether’s Theorem for
\textit{SMOOTH, DISCRETE} and
Finite Element Models

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Noether’s Theorem

links SYMMETRIES and conservation laws for Euler Lagrange Systems.

What is a conservation law?
Answer: a divergence expression which is zero on solutions of the system.

The heat equation

\[ u_t + (-u_x)_x = 0 \]

is its own conservation law. Integrating,

\[ \frac{\partial}{\partial t} \int_{\Omega} u + (-u_x) \bigg|_{\partial \Omega} = 0 \]

where we assume \( u \) is sufficiently nice that we can interchange \( \partial_t \) and \( \int \), and we have applied Stokes’ Theorem. In words:

Rate of change of total heat in \( \Omega \) = Net of comings and goings across the boundary
no sources or sinks
The usual examples

Symmetry leaves $Ldx$ invariant

Conserved Quantity
the quantity behind
the $\frac{D}{Dt}$
in the Divergence

\[
\begin{align*}
\begin{cases}
t^* &= t + c \\
\text{translation in time}
\end{cases} \\
\begin{cases}
x^*_i &= t + c \\
\text{translation in space}
\end{cases} \\
\begin{cases}
x^* &= \mathcal{R}x \\
\text{rotation in space}
\end{cases} \\
\begin{cases}
a^* &= \phi(a, b) \\
b^* &= \psi(a, b)
\end{cases}
\phi_a \psi_b - \phi_b \psi_a \equiv 1 \\
\text{Particle relabelling}
\end{align*}
\]
Variational Complexes 1-2-3!
are locally exact

**SMOOTH** cf. P.J. Olver, Applications . . .
\[
\begin{align*}
\text{Curl} & \quad \Lambda^2 \quad \text{Div} \quad \Lambda^3 \quad \hat{d} \quad \Lambda_1 \quad \hat{d} \quad \Lambda_2 \quad \hat{d} \\
\downarrow \pi & \quad \downarrow \pi \\
\Lambda^*_1 & \quad \delta \quad \Lambda^*_2 & \quad \delta
\end{align*}
\]

**DISCRETE** Hydon and ELM, J. FoCM
\[
\begin{align*}
\bigtriangleup & \quad \mathsf{Ex}^2 \quad \bigtriangleup \quad \mathsf{Ex}^3 \quad \hat{d} \quad \Lambda_1 \quad \hat{d} \quad \Lambda_2 \quad \hat{d} \\
\downarrow \pi & \quad \downarrow \pi \\
\Lambda^*_1 & \quad \delta \quad \Lambda^*_2 & \quad \delta
\end{align*}
\]

**Finite Element** ELM and GRW Quispel, CRM Proc.
\[
\begin{align*}
\downarrow & \quad \mathcal{F}^2 \quad \downarrow \quad \mathcal{F}^3 \quad \hat{d} \circ f \quad \mathcal{F}_1 \quad \hat{d} \quad \mathcal{F}_2 \quad \hat{d} \\
\downarrow \pi & \quad \downarrow \pi \\
\mathcal{F}^*_1 & \quad \delta \quad \mathcal{F}^*_2 & \quad \delta
\end{align*}
\]
Exactness can be used to find conservation laws for non Euler-Lagrange systems via clever ansatze!


Exactness is proved by the use of so-called homotopy operators $H_i$,

$$\begin{array}{ccc}
\text{Div} & \Lambda^2 & \Lambda^3 \\
\leftarrow & \leftarrow & \leftarrow \\
H_1 & H_1 & H_0 \\
\end{array}$$

which satisfy

$$(\text{Div}H_1 + H_0E)\omega = \omega, \quad \text{all } \omega \in \Lambda^3$$

Thus if $E(\omega) = 0$, then $\omega = \text{Div}(H_1(\omega)).$

Idea: solve $E($clever ansatz$) = 0$ for parameters and arbitrary functions. Then you have a conservation law using $H_1$. 
More on $\hat{d}$ and $\pi$

**SMOOTH**

$$\hat{d}(Ldx) = \hat{d}\left(\frac{1}{2}\left(u_x^2 + u_{xx}^2\right)dx\right) = (u_xdu_x + u_{xx}du_{xx})dx = (-u_{xx}du + u_{xxxx}du)dx + \frac{D}{Dx} (u_xdu - 2u_{xx}du_x + \frac{D}{Dx} (u_{xx}du)) = E(L)dudx + \frac{D}{Dx}\eta_L$$

General Formula, explicit, exact, symbolic, for $\eta_L$ known.

$$E = \pi \circ \hat{d}, \text{ where } \pi \text{ projects out the divergence term.}$$

More than one dependent variable

$$\hat{d}L(x, u, v, \ldots)dx = E^u(L)dudx E^v(L)dvdx + \frac{D}{Dx}\eta_L$$
More on $\hat{d}$ and $\pi$

**DISCRETE**

\[
\hat{d}(Ldx) = \hat{d}\left(\frac{1}{2}u_n^2 + uu_{n+1}\right)\Delta_n
\]

\[
= (u_n du_n + u_{n+2}du_n + u_n du_{n+2})\Delta_n
\]

\[
= (u_n + u_{n+2} + u_{n-2})du_n\Delta_n
\]

\[
+ (S - \text{id})(\ldots)
\]

\[
= E(L_n)du_n\Delta_n + \Delta(\eta_{L_n})
\]

General Formula, explicit, exact, symbolic, for $\eta_{L_n}$ known.

$E = \pi \circ \hat{d}$, where $\pi$ projects out the total difference term.

More than one dependent variable

\[
\hat{d}(L_n\Delta_n) = E^u(L_n)du_n\Delta_n + E^v(L_n)dv_n\Delta_n
\]

\[
+ \Delta(\eta_{L_n})
\]
Variational Symmetries

Symmetries arise from Lie group actions.

**EXAMPLE:** $G = (\mathbb{R}, +)$

$\epsilon \cdot x = x^* = \frac{x}{1 - \epsilon x}$, \hspace{1em} $\epsilon \cdot u = u^*(x^*) = \frac{u(x)}{1 - \epsilon x}$

Group Action Property

\[ \delta \cdot (\epsilon \cdot x) = \delta \cdot \left( \frac{x}{1 - \epsilon x} \right) = \frac{x}{1 - \delta \frac{x}{1 - \epsilon x}} \]

\[ = \frac{x}{1 - (\epsilon + \delta) x} = (\epsilon + \delta) \cdot x \]

and similarly for $u(x)$.

Prolonged Group Action

$\epsilon \cdot u_x = u_x^* = \frac{\partial u^*(x^*)}{\partial x} \frac{\partial x^*}{\partial x} = \frac{u_x}{(1 - \epsilon x)^2}$

and

$\delta \cdot (\epsilon \cdot u_x) = \frac{\delta \cdot u_x}{(1 - \epsilon (\delta \cdot x))^2} = \frac{u_x}{(1 - (\delta + \epsilon) x)^2}$
Action on Integrals

\[ \epsilon \cdot \int_{\Omega} L(x, u, u_x, \ldots) \, dx \]

def’n of \( \epsilon \cdot \)

\[ = \int_{\epsilon \cdot \Omega} L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \ldots) \, d\epsilon \cdot x \]

change of variable

\[ = \int_{\Omega} L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \ldots) \frac{d\epsilon \cdot x}{dx} \, dx \]

Use \( L^2 \) theory to get that a variational symmetry of a Lagrangian is a group action such that

\[ L(x, u, u_x, \ldots) = L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \ldots) \frac{d\epsilon \cdot x}{dx} \]
Infinitesimal Action on Integrals

Since the symmetry invariance condition
\[ L(x, u, u_x, \ldots) = L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \ldots) \frac{d\epsilon \cdot x}{dx} \]
is true all \( \epsilon \), then if everything is sufficiently smooth, applying \( \frac{d}{d\epsilon}\big|_{\epsilon=0} \) to both sides, and noting that when \( \epsilon = 0 \) we have the identity action,

\[
0 = \frac{\partial L}{\partial x} \xi + \frac{\partial L}{\partial u} \phi + \frac{\partial L}{\partial u_x} \phi^x + \cdots + L \xi_x
\]

\[ = \frac{D(L \xi)}{Dx} + \frac{\partial L}{\partial u} Q + \frac{\partial L}{\partial u_x} DQ + \frac{\partial L}{\partial u_{xx}} D^2Q + \cdots \]

where

\[ Q = \phi - u_x \xi, \quad \phi = \frac{d}{d\epsilon}|_{\epsilon=0} \epsilon \cdot u, \quad \xi = \frac{d}{d\epsilon}|_{\epsilon=0} \epsilon \cdot x \]

and \( \frac{D}{Dx} \) is the total derivative operator.

\[
0 = \text{Div}(L \xi) + \sum (D^J Q^\alpha) \frac{\partial L}{\partial u^\alpha_J}
\]
Almost to the punchline

Let

\[ \mathbf{v}_Q = \sum_{\alpha} Q^\alpha \frac{\partial}{\partial u^\alpha} \]

Then the \textit{prolongation} is defined by

\[ \text{pr}\mathbf{v}_Q = \sum_{\alpha, J} D^J Q^\alpha \frac{\partial}{\partial u^\alpha} \]

Note

\[ u^\alpha_J = \frac{\partial u^\alpha}{\partial x^1_{J_1} \ldots \partial x^p_{J_p}} = D^J u^\alpha \]

Then

\[ \sum \left( D^J Q^\alpha \right) \frac{\partial L}{\partial u^\alpha_J} = \text{pr}\mathbf{v}_Q \hat{\partial} L \]

Recall that \( \hat{\partial} \) is one of the two operators comprising the Euler Lagrange operator, while the left hand side is a divergence if \( Q \) is the characteristic of a symmetry.
THE PUNCHLINE

\[ \ldots \text{Curl} \quad \Lambda^2 \xrightarrow{\text{Div}} \quad \Lambda^3 \xrightarrow{\psi} \quad E \xrightarrow{\Lambda^1 \pi^\ast \hat{d}} \ldots \]

\[ \psi_{L} \]

\[ E^\alpha(L) du^\alpha dx \]

\[ = \hat{d}(L) + \text{Div}(\eta_{L}) \]

\[ \text{pr} \mathbf{v}_{Q \perp} \]

\[ Q \cdot E(L) \]

\[ = \mathbf{v}_{Q \perp} \hat{d}(L) + \text{Div}(\text{pr} \mathbf{v}_{Q \perp} \eta_{L}) \]

If \( Q \) is the characteristic of a symmetry, we have that

\[ \mathbf{v}_{Q \perp} \hat{d}(L) = \text{Div}(L \xi) \]

and hence that

\[ Q \cdot E(L) = \text{Div}(\text{something}) \]
Non-trivial example

Semi-geostrophic equations

Group
\[
\begin{align*}
    a^* &= \phi(a, b) \\
    b^* &= \psi(a, b)
\end{align*}
\]

Invariants
\[
\begin{align*}
    h &= (x_a y_b - x_b y_a)^{-1} \\
    \partial_x &= h(y_b \partial_a - y_a \partial_b) \\
    \partial_y &= h(-x_b \partial_a + x_a \partial_b)
\end{align*}
\]

Equations
\[
\begin{align*}
    D_t x &= -\frac{g}{f^2} D_t h_x - \frac{g}{f} h_y \\
    D_t y &= -\frac{g}{f^2} D_t h_y + \frac{g}{f} h_x
\end{align*}
\]

The Lagrangian has 4 arbitrary functions which obey two conditions. The conserved quantity is potential vorticity

\[
\frac{1}{h} \left( f + \frac{g}{f} (h_{xx} + h_{yy}) \frac{g^2}{f^3} (h_{xx} h_{yy} - h_{xy}^2) \right)
\]
DISCRETE Almost Punchline

This case is easier than the smooth case.

- Since \( n \) cannot vary in a smooth way, the “mesh variables” \( x_n \) are treated as dependent variables.

- The group action commutes with shift:
\[
\epsilon \cdot S^j(u_n) = \epsilon \dot{u}_{n+j} = S^j \epsilon \cdot u_n
\]
so no prolongation formulae are required.

For example,
\[
\epsilon \cdot u_n = \frac{u_n}{1 - \epsilon x_n} \implies \epsilon \cdot u_{n+j} = \frac{u_{n+j}}{1 - \epsilon x_{n+j}}
\]

The symmetry condition is:
\[
L_n(x_n, \ldots, x_{n+j}, u_n, \ldots, u_{n+k}) = L_n(x^*_n, \ldots, x^*_{n+j}, u^*_n, \ldots, u^*_{n+k})
\]
where \((\cdot)^* \equiv \epsilon \cdot (\cdot)\).
Applying
\[
\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} = 0
\]
to both sides of the symmetry condition yields
\[
0 = \sum_k \frac{\partial L_n}{\partial x_n+k} \frac{d}{d\epsilon} \left|_{\epsilon=0} \right. x_n^*+k + \frac{\partial L_n}{\partial u_n+k} \frac{d}{d\epsilon} \left|_{\epsilon=0} \right. u_n^*+k
\]
Setting
\[
Q_n^x = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} x_n^*, \quad Q_n^u = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} u_n^*
\]
then since
\[
Q_{n+k}^x = S^k(Q_n^x), \quad Q_{n+k}^u = S^k(Q_n^u)
\]
the equation above can be written as
\[
0 = X_Q \hat{d} L_n, \quad X_Q = \sum_{\alpha,J} S^J(Q_n^\alpha) \frac{\partial}{\partial u_{n+J}^\alpha}
\]
DISCRETE Punchline

\[ \Lambda \rightarrow \mathbf{Ex}^2 \rightarrow \Lambda^1 \rightarrow \Lambda^2 \]
\[ \Xi \rightarrow \mathbf{Ex}^3 \rightarrow \Psi \rightarrow \mathbf{E}(L_n) \, \mathbf{d}u_n \]
\[ = \mathbf{d}(L_n) + \Delta(\eta_{L_n}) \]
\[ \leftarrow X_{Q-} \]

\[ Q \cdot E(L_n) = X_{Q-} \mathbf{d}(L_n) + \Delta(X_{Q-} \eta_{L_n}) \]

Again, we get that if

\[ X_{Q-} \mathbf{d}(L_n) = 0 \]

then

\[ Q \cdot E(L_n) = \Delta(\text{something}), \]

that is, a total difference expression which is zero on solutions of the discrete Euler Lagrange system.
A difference model for $\int (\frac{1}{2}\dot{x}^2 - V(x)) \, dt$

Define

$$\overline{V}(n) = \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} V(x) \, dx$$

and take

$$L_n = \left[ \frac{1}{2} \left( \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right)^2 - \overline{V}(n) \right] (t_n - t_{n-1})$$

The group action is $t^*_n = t_n + \epsilon$, with $x_n$ invariant. The conserved quantity is thus "energy". Now, $Q^t_n = 1$ for all $n$, and $Q^x_n = 0$. The equations become

$$0 = E^t(L_n) = \frac{\partial}{\partial t_n} L_n + S \left( \frac{\partial}{\partial t_{n-1}} L_n \right)$$

$$0 = X_{Q^x} L_n = \frac{\partial}{\partial t_n} L_n + \frac{\partial}{\partial t_{n-1}} L_n$$

as $L_n$ is a function of $(t_n - t_{n-1})$. 

It is easy to see in this case that

$$0 = (S - \text{id}) \left( \frac{\partial}{\partial t_n} L_n \right)$$

is implied by the two equations, to yield

$$\frac{1}{2} \left( \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right)^2 + \bar{V}(n) = c$$

Note that the energy in the smooth case is

$$1/2 \dot{x}^2 + V.$$ 

Can regard the EL eqn for the mesh variables as an equation for a variable mesh.
INTERLUDE

If we know the group action for a particular conservation law, we can “design in” that conservation law into a discretisation by taking a Lagrangian composed of invariants. These necessarily satisfy $v_Q(I) = 0$ or $X_Q(I_n) = 0$. The Fels and Olver formulation of moving frames is particularly helpful here: a sample theorem is

Discrete rotation invariants in $\mathbb{Z}^2$

Let $(x_n, y_n), (x_m, y_m)$ be two points in the plane. Then

$$I_{n,m} = x_n y_n + x_m y_m, \quad J_{n,m} = x_n y_m - x_m y_n$$

are rotation invariants. Moreover, any discrete rotation invariant is a function of these.
Made up example

Suppose

\[ L_n = \frac{1}{2} J_{n,n+1}^2 = \frac{1}{2} (x_n y_{n+1} - x_{n+1} y_n)^2 \]

then

\[
\begin{align*}
E_n^x &= J_{n,n+1} y_{n+1} - J_{n-1,n} y_{n-1} \\
E_n^y &= -J_{n,n+1} x_{n+1} + J_{n-1,n} x_{n-1}
\end{align*}
\]

Now,

\[ Q_n = (Q_n^x, Q_n^y) = (-y_n, x_n) = \left. \frac{d}{d\theta} \right|_{\theta=0} (x_n^*, y_n^*) \]

and thus

\[ Q_n \cdot E_n = J_{n,n+1} (-y_n y_{n+1} - x_n x_{n+1}) + J_{n-1,n} (y_n y_{n-1} + x_n x_{n-1}) = -J_{n,n+1} I_{n,n+1} + J_{n-1,n} I_{n-1,n} = -(S - \text{id}) (J_{n-1,n} I_{n-1,n}) \]

gives the conserved quantity.

Note that \( I_{n,m} = I_{m,n} \) and \( J_{n,m} = -J_{m,n} \).
Less easy example

Hereman et al., Densities, Symmetries and Recursion operators for nonlinear DDEs, CRM Proceedings

The Toda lattice in polynomial form is

\[
\begin{align*}
\dot{u}_n &= v_{n-1} - v_n \\
\dot{v}_n &= v_n(u_n - u_{n+1})
\end{align*}
\]

The scaling symmetry is the basis for the ansatz used to obtain the differential-difference conservation laws, which are of the form

\[
\frac{D}{Dt}\rho_n + (S - \text{id})J_n = 0
\]

for example

\[
\rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} - v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2
\]

These results use the ansatz plus homotopy operator method outlined earlier.
Summary of the Pattern

\[
\begin{array}{c}
\{ \text{Div} \} \\
\Delta \\
\rightarrow \\
\{ \Lambda^3 \} \\
\psi \\
\{ \text{Ex}^3 \} \\
L \\
\end{array}
\xrightarrow{\pi \circ \tilde{d}} \\
\Lambda^1 \\
\rightarrow \\
\Lambda^2 \\
\rightarrow \\
L \\
\sum_\alpha E^\alpha(L) d\nu^\alpha \\
= \tilde{d}(L) + \left\{ \begin{array}{c}
\text{Div} \\
\Delta \\
\end{array} \right\} \eta_L
\]

\[
Q \cdot E(L) = \left\{ \begin{array}{c}
v_Q \\
X_Q \\
\end{array} \right\} \tilde{d}L + \left\{ \begin{array}{c}
\text{Div} \\
\Delta \\
\end{array} \right\} \left\{ \begin{array}{c}
v_Q \\
X_Q \\
\end{array} \right\} \eta_L
\]

- the formula for \( \eta_L \) is explicit, exact, symbolic
- the first summand is a total derivative or difference by the symmetry condition
OK let’s try for a Neother’s Theorem for Finite Element!

D. Arnold, Beijing ICM Plenary talk

**Given** a system of moments and sundry other data, aka degrees of freedom, that yield projection operators such that the diagram commutes:

\[
0 \to \mathbb{R} \to \Lambda^0 \to \Lambda^1 \to \Lambda^2 \to \Lambda^3 \to 0
\]

\[
\Pi_0 \downarrow \quad \Pi_1 \downarrow \quad \Pi_2 \downarrow \quad \Pi_3 \downarrow
\]

\[
0 \to \mathbb{R} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \mathcal{F}^3 \to 0
\]

all relative to some triangulation.

**Yields stability!!** A Lagrangian is composed of wedge products of 1-, 2- and 3- forms. Choose the discretisation of each to be in the relevant \( \mathcal{F}_i \). Then commutativity implies conditions for Brezzi’s theorem to hold.
In one dimension: with \( e_n = (x_n, x_{n+1}) \), \( \Pi_0 \) to piecewise linear, \( \Pi_1 \) to piecewise constant with moment

\[
\alpha_n = \int_{x_n}^{x_{n+1}} u(x) \psi_n(x) \, dx
\]

Commutativity of the diagram

\[
\begin{array}{c}
\Pi_0 \downarrow \\
\downarrow \Pi_1
\end{array}
\]

\[
u|_{e_n} = A_n x + B_n \quad \Rightarrow \quad A_n = \int_{x_n}^{x_{n+1}} u'(x) \psi_n(x) \, dx
\]

implies

\[
A_n = u(x) \psi_n(x) \bigg|_{x_n}^{x_{n+1}} - \int_{x_n}^{x_{n+1}} u(x) \psi_n'(x) \, dx
\]

Note that

\[
\int_{x_n}^{x_{n+1}} \psi_n(x) \, dx = 1.
\]

is required by the projection property.
A finite element Lagrangian is built up of wedge products of forms in $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. Call this resulting space $\tilde{\mathcal{F}}_3$. In each top-dimensional simplex, denoted $\tau$, integrate to get

$$L = \sum_{\tau} L_\tau(\alpha^1_\tau, \cdots \alpha^p_\tau)$$

where $\alpha^j_\tau$ is the $j^{th}$ degree of freedom in $\tau$. $L$ can also depend on mesh data.

Can now take $\hat{d}$ which is the variation with respect to the $\alpha^j_\tau$. 
**Example** In one dimension,

\[ 0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \to 0 \]
\[ \Pi_0 \downarrow \quad \Pi_1 \downarrow \]
\[ 0 \to \mathbb{R} \to \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1 \to 0 \]

\( \Pi_1 \) is to piecewise constant functions with moment \( \bar{u}(n) = \int_{x_n}^{x_{n+2}} u(x) \psi_n(x) \, dx \) where

![Diagram](image.png)

on \( (x_n, x_{n+2}) \), while \( \Pi_0 \) is to piecewise linear functions with moments

\[ \alpha_n = \frac{1}{x_{n+2} - x_n} \int_{x_n}^{x_{n+1}} u(x) \, dx \]

that is, \( \alpha_n, \alpha_{n+1} \) are used in \( (x_n, x_{n+2}) \);

\[ u \mapsto 2\frac{\alpha_{n+1} - \alpha_n}{x_{n+2} - x_n} x + \left( \frac{x_{n+1} + x_{n+2}}{x_{n+2} - x_n} \right) \alpha_n - \left( \frac{x_{n+1} + x_n}{x_{n+2} - x_n} \right) \alpha_{n+1} \]
Very simple example

\[ \int \frac{1}{2} u_x^2 \, dx \] projects to

\[ \sum_n \int_{x_{2n}}^{x_{2n+2}} \frac{1}{2} (u_x)^2 \, dx = \sum_n 2 \left( \frac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} \right)^2. \]

Then

\[ \hat{d}L_{2n} = 4 \frac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} (d\alpha_{2n} - d\alpha_{2n+1}) \]

\[ = 4 \left( \frac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} - \frac{\alpha_{2n-1} - \alpha_{2n}}{x_{2n+1} - x_{2n}} \right) d\alpha_{2n} \]

\[ + (S - \text{id})(\text{something}) \]

The discrete Euler Lagrange equation is then, after "integration",

\[ \frac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} = c \]
Look now at the “Noether pattern” for the Finite Element variational complex

\[ \rightarrow \tilde{\mathcal{F}}_2 \rightarrow \tilde{\mathcal{F}}_3 \xrightarrow{\pi \circ \delta \circ f} \tilde{\mathcal{F}}^1 \rightarrow \tilde{\mathcal{F}}^2 \rightarrow \bigcup_{L_T} \tilde{\mathcal{F}}^* \rightarrow \bigcup_{E(L_T) + \delta(\eta_L)} \tilde{\mathcal{F}}^* \rightarrow \tilde{\mathcal{F}}_\tau \rightarrow \bigcup_{\delta L_T} \tilde{\mathcal{F}}^*_\tau \rightarrow \bigcup_{v_{Q_T}} \delta(\tilde{\mathcal{F}}_\tau) = \delta(\text{something}) \]

where \( \delta \) is the mesh dependent coboundary operator (recall \( \delta(f)(\tau) = f(\partial \tau) \)).

**Step 1:** find \( \eta_L \) \hspace{1cm} **Step 2:** find \( v_Q \)

If then \( v_{Q_T} \cup \delta(L_T) = \delta(\text{something}) \) we will have that

\[ 0 = Q_\tau \cdot E(L_T) + \delta(\text{something}). \]
Group actions on moments

The clue is the variational symmetry group action on $\int_{\Omega} L(x, u, \cdots) \, dx$

Define

$$\epsilon \cdot \int_{\tau} u(x) \psi_{\tau}(x) \, dx$$

$$= \int_{\tau} \epsilon \cdot u(x) \psi_{\tau}(\epsilon \cdot x) \frac{d(\epsilon \cdot x)}{dx} \, dx$$

**Example** Recall the projective action

$$\epsilon \cdot x = \frac{x}{1 - \epsilon x}, \quad \epsilon \cdot u(x) = \frac{u(x)}{1 - \epsilon x}$$

Then the induced action on the moments

$$\alpha_n = \int_{x_n}^{x_{n+1}} \frac{u(x)}{x^3} \, dx, \quad \beta_n = \int_{x_n}^{x_{n+1}} \frac{u(x)}{x^4} \, dx$$

is

$$\epsilon \cdot \alpha_n = \alpha_n, \quad \epsilon \cdot \beta_n = \beta_n - \epsilon \alpha_n$$
In general for this action,

\[ \epsilon \cdot \int_{x_n}^{x_{n+1}} x^m u(x) \, dx \]

\[ = \int_{x_n}^{x_{n+1}} \frac{x^m}{(1-\epsilon x)^m} \frac{u(x)}{1-\epsilon x} \frac{dx}{(1-\epsilon x)^2} \]

\[ = \int_{x_n}^{x_{n+1}} \frac{x^m u(x)}{(1-\epsilon x)^{m+2}} \, dx \]

THINK: if you want a coherent scheme which maps to itself under this projective action, and involves only a finite amount of data, then take your moments to be

\[ u(x) \mapsto \int_{x_n}^{x_{n+1}} \frac{u(x)}{x^m} \, dx, \quad m = 3, 4, \ldots N. \]
CONCLUSIONS

- The underlying algebraic pattern of the exact variational complexes provide a framework for generalisations of Noether’s Theorem and conservation laws in general.

- Symmetry-adapted moments would appear to be necessary.

- Next: formulae for $\eta_{L_T}$ where

$$\hat{d}(L_T) = E(L_T) + \delta(\eta_{L_T})$$

in terms of the mesh dependent coboundary operator.